

FISICA del PLASMA e MATEMATICA APPLICATA

(aula Marconi, CNR, Roma – 06 Giugno 2018)

PROGRAMMA

1. h. 10.00: “Basic Theoretical Issues Raised by the Physics of High Energy Plasmas (in Astrophysics and Laboratory)”, Bruno Coppi, MIT.
2. h. 10.30: “A microscopic model for uphill diffusion”, Matteo Colangeli, Università de L’Aquila.
3. h. 11.00: “Spectral analysis for isogeometric discretizations of curl-div problems”, Carla Manni, Università di Roma - Tor Vergata.
4. h. 11.30: “Inflexional instability and twist bursts of magnetic fields”, Renzo Ricca, Università di Milano–Bicocca.
5. h. 12.00: “Multiscale numerical methods in plasma physics”, Giacomo Dimarco, Università di Ferrara.
6. h. 12.30: “Inverse problems for solar plasma spectroscopy, imaging and imaging spectroscopy”, Michele Piana, Università di Genova.

Interruzione: h. 13.00 – 14.30.

7. h. 14.30: “Lagrangian Coherent Structures in magnetized plasmas”, Daniela Grasso, CNR, Torino.
8. h. 15.00: “Validazione di modelli di tipo Vlasov-(Navier)-Stokes per sistemi bifase”, Valeria Ricci, Università di Palermo.
9. h. 15.30: “Costruzione di un modello di Maxwell-Vlasov per la propagazione e l’assorbimento di onde elettromagnetiche in plasmi confinati magneticamente”, Alessandro Cardinali, ENEA–Frascati.

(segue)

10. h. 16.00: “Soluzioni innovative per il calcolo numerico di campi elettromagnetici in configurazioni d’interesse per la fusione nucleare”, Guglielmo Rubinacci, Università di Napoli.
11. h. 16.30 “Magnetic reconnection sustained by the thermonuclear heating of the electron population,” Renato Gatto, Sapienza Università di Roma.

Basic Theoretical Issues Raised by the Physics of High Energy Plasmas

CNR, 6 June 2018

Major new developments that have led to growth of plasma physics into an independent and sophisticated field of research:

- Space Physics: Exploration of the Heliosphere and in situ investigations of the Earth's magnetosphere
- High Energy Astrophysics and Radio Astronomy (c.c. beyond the visible)
- Controlled Nuclear Fusion Research (preceded by and in parallel to “Uncontrolled Fusion”)

These have led to deal with a wide variety of issues that have required the formulation of new mathematical problems and the efforts to solve them

- Stationary plasma configurations (e.g. mirror, axisymmetric and non-axisymmetric toroidal equilibrium configurations)
- Macroscopic plasma instabilities
- Microscopic and mesoscopic plasma collective modes
- Transport processes (e.g. electron and ion thermal energy, particle and angular momentum transport) due to collective modes
- Rotating Plasma Features

- Magnetic Reconnection Processes: destroying and producing magnetic fields
- Collisionless Shocks and High Energy Particle Acceleration Processes
- Plasmas associated with single black holes or with binaries of black holes and of other collapsed objects
- “G.R. microscopic scale distances” and shorter scale distances. Bright spots.
- Gravitational Plasmas and E.M. Interactions

- Limits of Fluid descriptions and necessity of formulation involving geometry and momentum space. Example: The Collisionless Trapped Electron Mode (TEM)
- Unsolved integral equation for Ion Temperature Gradient (ITG) driven modes also relevant to advanced confinement experiments
- Slide-away and other non-thermal regimes
- High Energy Radiation Emission Processes from Collapsing Binaries.

Instabilities due to Temperature Gradients in Complex Magnetic Field Configurations

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(Received 25 July 1966)

An integral equation governing an instability due to ion temperature gradients is derived. In the presence of magnetic shear, localized non-convective normal modes of instability are shown to exist if the relative temperature gradient is larger than that of density, unless the shear is exceedingly strong, i.e., the field shears through a large angle in the distance in which the temperature drops. Quasi-modes which are less localized in the direction of the gradient can be constructed from these normal modes and a large thermal diffusion may be expected. Conversely the mass diffusion is shown to be rather slow so that it is reasonable to assume that an effective "divertor" should keep the actual heat loss quite small.

I. INTRODUCTION

THE effects of drift instabilities due to temperature gradients transverse to the magnetic field lines in complex magnetic field configurations, such as in the presence of shear, have recently become a subject of interest.¹ The reason is that relevant modes are not stabilized by ion Landau damping. In the present work we show that the introduction of magnetic shear in the equilibrium actually gives rise to unstable normal modes which are non-convective,² and may be strongly localized in space. These modes are expected to arise in the neighbourhood of the plasma container wall where the temperature gradient is higher than the density gradient, and may seriously contribute to diffusion of and thermal leakage from the contained plasma.

II. MODEL

We consider a one-dimensional configuration having low β with density and temperature depending on x . The main magnetic field is assumed to be along the z direction. The magnetic shear is represented by a small component along the y direction, so that $\mathbf{B} = B_0(\mathbf{e}_z + x/L_s\mathbf{e}_y)$. No electric field exists in the equilibrium so that the equilibrium distribution function can be taken as of the form³

$$f_0 = n(m/2\pi T)^{3/2} e^{-mv^2/2T}.$$

Here n and T are assumed functions of $x + v_y/\Omega$, Ω being the gyrofrequency. Then we consider electrostatic perturbations so that $\mathbf{E} = -\nabla\phi$ with normal mode solutions of the form $\phi = \phi_k(x) \exp(i\omega t + ik_y y + ik_z z)$. We use the Vlasov equation,

¹ B. B. Kadomtsev, in *Plasma Physics and Controlled Nuclear Fusion Research* (International Atomic Energy Agency, Vienna, 1966), Vol. 1, p. 29.

² B. Coppi, G. Laval, R. Pellat, and M. N. Rosenbluth Nucl. Fusion (to be published).

³ M. N. Rosenbluth, in *Plasma Physics* (International Atomic Energy Agency, Vienna, 1965), p. 501.

integrating the perturbed linearized form of it along particle orbits. In particular, we are interested in the case $\omega \approx k_{\parallel} v_{thi} \approx k_y v_{di}$ and $k_{\perp} a_i \approx 1$, where v_{thi} is the ion thermal velocity and v_{di} is the ion diamagnetic velocity, and a_i the ion gyroradius. Therefore,

$$\frac{df}{dt} = -\frac{e}{m} \left(2\mathbf{E} \cdot \nabla \frac{\partial f}{\partial v^2} + E_y \frac{\partial}{\partial v_y} f_0 \right),$$

$$f = -\frac{e}{T} \left[\phi - i \left(\omega + \frac{k_y T}{\Omega m} \frac{\partial}{\partial x} \right) \right] f_0 \int_{-\infty}^t \phi(\mathbf{x}') dt',$$

x' and t' referring to equilibrium particle trajectories. The x and y components of these are in fact

$$x' = x + (v_{\perp}/\Omega)[\sin(\Omega t' + \psi) - \sin \psi],$$

$$y' = y + (v_{\perp}/\Omega)[\cos(\Omega t' + \psi) - \cos \psi]$$

$$+ (v_{\parallel} t'/L_s)[x - (v_{\perp}/\Omega) \sin \psi].$$

Introducing the space Fourier transform

$$\phi_k(x) = \frac{1}{(2\pi)^{1/2}} \int e^{-ik'x} \phi(k') dk'$$

we have, referring to the ion perturbed density

$$\tilde{n}_i(k) = -\frac{ne}{T} \phi(k) + \frac{e}{T} \left(\omega + \frac{k_y T}{\Omega M} \frac{\partial}{\partial x} \right) n \left(\frac{M}{2\pi T} \right)^{3/2} \frac{1}{2\pi}$$

$$\cdot \int \exp \left(\frac{Mv^2}{2T} \right) d^3\mathbf{v} \int dx \int dk' \phi(k')$$

$$\cdot \int_{-\infty}^0 dt' \exp(i\omega t') \exp[i(k - k')x]$$

$$\cdot \exp \left[-ik' \frac{v_{\perp}}{\Omega} \sin(\Omega t' + \psi) \right]$$

$$\cdot \exp \left(-ik' \frac{v_{\perp}}{\Omega} \sin \psi \right)$$

$$\cdot \exp \left[ik_y v_{\parallel} \frac{t'}{L_s} \left(x - \frac{v_{\perp}}{\Omega} \sin \psi \right) \right]$$

$$+ ik_y \left(\frac{v_{\perp}}{\Omega} \cos(\Omega t' + \psi) - \frac{v_{\perp}}{\Omega} \cos \psi \right) \Big]. \quad (1)$$

More specifically, we can write the quantity on which $\partial/\partial x$ operates as

$$\begin{aligned} \mathfrak{I} = & n \left(\frac{M}{2\pi T} \right)^{\frac{1}{2}} \int_0^{2\pi} d\psi \int_0^{+\infty} v_{\perp} dv_{\perp} \exp \left(-M \frac{v_{\perp}^2}{2T} \right) \\ & \cdot \int_{-\infty}^{+\infty} dv_{\parallel} \exp \left(-\frac{M v_{\parallel}^2}{2T} \right) \int \phi(k') dk' \int_{-\infty}^0 dt' \\ & \cdot \int dx \exp \left[i \left(x - \frac{v_{\perp}}{\Omega} \sin \psi \right) \left(k - k' + ik_v \frac{v_{\parallel} t'}{L_s} \right) \right] \\ & \cdot \exp \left(ik \frac{v_{\perp}}{\Omega} \sin \psi \right) \exp \left(-ik' \frac{v_{\perp}}{\Omega} \sin (\Omega t' + \psi) \right) \\ & \cdot \exp \left(ik_v \frac{v_{\perp}}{\Omega} \cos (\Omega t' + \psi) \right) \exp (i\omega t'). \end{aligned}$$

If we carry out integration over $v_{\parallel}$, $x - (v_{\perp}/\Omega) \sin \psi$, and $v_{\perp}$ neglecting terms of order ω/Ω in comparison with 1, we have

$$\begin{aligned} \mathfrak{I} = & n \frac{(2\pi)^{\frac{1}{2}}}{k_v} L_s \left(\frac{M}{T} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} dk' \phi(k') \\ & \cdot \int_{-\infty}^0 dt' \exp (i\omega t') \exp \left(-\frac{1}{2} (k - k')^2 \frac{M}{T} \frac{L_s^2}{k_v^2 t'^2} \right) \\ & \cdot \exp \left(-\frac{T}{2M\Omega^2} (k^2 + k'^2 + 2kk') \right) \\ & \cdot I_0 \left((k^2 + k_v^2)^{\frac{1}{2}} (k'^2 + k_v^2)^{\frac{1}{2}} \frac{T}{M\Omega^2} \right). \end{aligned} \quad (2)$$

Since we are interested in perturbations having phase velocity along the magnetic field less than the electron thermal velocity and on which the electron Landau damping is not important (see Sec. III) we take $\tilde{n}_e = e\phi(k)/T_e$. Then introducing the dimensionless units $t = t'\omega$,

$$\bar{k} = (T/M)^{\frac{1}{2}} \Omega^{-1} k = (1/\sqrt{2}) a_i k,$$

a_i being the ion Larmor radius, and

$$\bar{\omega} = \frac{\omega}{k_v v_{di}} = \omega \frac{\Omega M}{T} \left| \frac{T}{T'} \right| \frac{1}{k_v},$$

v_{di} representing the ion diamagnetic velocity, we are led to consider the integral equation,

$$\begin{aligned} \phi(\bar{k}) \left(\frac{T_i}{T_e} + 1 \right) = & i \frac{L_s}{r_T} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} d\bar{k}' \phi(\bar{k}') \\ & \cdot \int_{-\infty}^0 dt \frac{e^{it}}{|t|} \exp \left(-\frac{1}{2} (\bar{k} - \bar{k}')^2 \frac{L_s^2}{r_T^2} \frac{\bar{\omega}^2}{t^2} \right) \\ & \cdot \left[\bar{\omega} - \left(\frac{r_T}{r_n} - \frac{1}{2} - \frac{1}{2} it - \frac{1}{2} (\kappa^2 + \kappa'^2) \right) \right. \\ & \left. + \kappa \kappa' \frac{I_1(\kappa \kappa')}{I_0(\kappa \kappa')} \right] I_0(\kappa \kappa') \exp \left[-\frac{1}{2} (\kappa'^2 + \kappa^2) \right], \end{aligned} \quad (3)$$

$$\text{where } \kappa = (b + \bar{k}^2)^{\frac{1}{2}}, \kappa' = (b + \bar{k}'^2)^{\frac{1}{2}},$$

$$b = \frac{1}{2} (k_v a_i)^2,$$

$$r_n = - \left(\frac{d \ln n}{dx} \right)^{-1}, \quad r_T = - \left(\frac{d \ln T}{dx} \right)^{-1}.$$

We have derived this equation in view of studying the localized modes due to temperature gradient, which are found in Sec. III in the fluid approximation for $T'/T \gg n'/n$ (strong temperature gradient) and $L_s \gg r_T$ (relatively weak shear). In particular we are interested in the critical value of r_T/L_s at which stability occurs. We know⁴ that the instability disappears when $n'/n > T'/T$. On the other hand for $T'/T > n'/n$ we shall see in Sec. III that stability cannot be achieved as long as $L_s \gg r_T$. To determine the exact critical value for L_s/r_T will require a numerical solution of Eq. (3) which will be undertaken in a later work. We note here only that for $L_s/r_T \ll 1$ no unstable eigenvalue may be found as the real part of the right-hand side becomes very small for all unstable $\bar{\omega}$ and cannot balance the left-hand side.

III. THE FLUID APPROXIMATION

The most immediate limit in which Eq. (3) can be solved corresponds to wavelengths longer than the ion Larmor radius so that $\bar{k} < 1$ and $b < 1$, and to neglecting the effects of ion Landau damping so that

$$\begin{aligned} \phi(\bar{k}') \approx & \phi(\bar{k}) + (\bar{k}' - \bar{k}) \frac{d\phi}{d\bar{k}} + \frac{1}{2} (\bar{k}' - \bar{k})^2 \frac{d^2\phi}{d\bar{k}^2} \\ \kappa'^2 \approx & \bar{k}^2 + b + 2\bar{k}(\bar{k}' - \bar{k}) + (\bar{k}' - \bar{k})^2. \end{aligned} \quad (4)$$

Then we have in lowest order

$$\begin{aligned} \left(1 + \frac{T_i}{T_e} \right) \phi(\bar{k}) = & -i \frac{L_s}{r_T \pi^{\frac{1}{2}}} \\ & \cdot \int_{-\infty}^0 dt \frac{\exp(it)}{|t|} \left(\left[\bar{\omega} - \left(\frac{r_T}{r_n} - \frac{1}{2} - \frac{1}{2} it - \bar{k}^2 - b \right) \right] \right. \\ & \cdot (1 - \bar{k}^2 - b) \phi(\bar{k}) \int_{-\infty}^{+\infty} d\zeta \exp \left(-\frac{1}{2} \zeta^2 \frac{L_s^2}{r_T^2} \frac{\bar{\omega}^2}{t^2} \right) \\ & - \left\{ \left[\bar{\omega} - \left(\frac{r_T}{r_n} - \frac{1}{2} - \frac{1}{2} it \right) - 1 \right] \bar{k} \frac{d\phi}{d\bar{k}} \right. \\ & \left. - \left[\bar{\omega} - \left(\frac{r_T}{r_n} - \frac{1}{2} - \frac{1}{2} it \right) \right] \frac{d^2\phi}{d\bar{k}^2} \right\} \\ & \left. \cdot \int_{-\infty}^{+\infty} d\zeta \exp \left(-\frac{1}{2} \zeta^2 \frac{L_s^2}{r_T^2} \frac{\bar{\omega}^2}{t^2} \right) \right), \end{aligned} \quad (5)$$

⁴ A. A. Galeev, V. N. Oraevskii and R. Z. Sagdeev, Zh. Eksperim. i Teor. Fiz. **44**, 903 (1963) [English transl.: Soviet Phys.—JETP **17**, 815 (1963)].

3. Twin Disks Configuration

The unperturbed state is characterized by a twin plasma disks configuration as described in the following. We assume, for the simplicity that this configuration is immersed in a constant vertical magnetic field $\mathbf{B} = B\mathbf{e}_z$ and that electrons and nuclei have the same longitudinal (high) temperature $T_{\parallel}$. Therefore, considering that $\varepsilon_k^2 \ll 1$, to lowest order in this parameter the configuration of interest is stationary. The relevant vertical equilibrium condition, ignoring the effect of the modulated potential Φ_G^0 , is

$$-\Omega_k^2 z \rho_h - \frac{2}{m_i} T_{\parallel} \frac{d\rho_h}{dz} \simeq 0, \quad (5)$$

where $\rho_h \simeq m_i n_h$ with $n_h \equiv n_{ih} = n_{eh}$ represents the density of the “hot” energy particle population (electrons and nuclei) with a temperature independent of z . Then

$$\rho_h(z, R_0) \simeq \rho_h^0 \exp\left(-\frac{z^2}{2H_h^2}\right), \quad (6)$$

as is well known, where

$$H_h \equiv \left(\frac{2}{m_i} \frac{T_{\parallel}}{\Omega_k^2}\right)^{1/2}$$

represents the height of the stationary disk that is sustained by Φ_G^0 .

Clearly, the one-fluid Eq. (5) implies that the nuclei are gravitationally confined while the electrons are electrostatically confined. In particular, for $\mathbf{E} = -\nabla\Phi_E$ we have, at

$$R = R_0,$$

$$-en_h \frac{\partial\Phi_E}{\partial z} + T_{\parallel} \frac{dn_h}{dz} = 0 \quad (7)$$

implying that

$$\Phi_E \simeq -\frac{T_{\parallel}}{e} \frac{z^2}{2H_h^2} + B \frac{\Omega_k R_0^2}{c} \ln\left(\frac{R_0}{R}\right) \quad (8)$$

as the radial equilibrium conditions $v_{\phi} = \Omega_k R$ leads to identify a radial electric field component given by

$$E_R + \frac{1}{c} v_\phi B_z \simeq 0. \quad (9)$$

The vertical momentum balance equation to next order in ε_k^2 can be written as

$$-z \left\{ \Omega_k^2 \hat{\rho}_f + \tilde{\Omega}_k^2 \rho_h \cos[2(\varphi - \Omega_{ob} t)] \right\} \simeq 2 \frac{T_\parallel}{m_i} \frac{\partial}{\partial z} \hat{\rho}_f \quad (10)$$

where $\hat{\rho}_f$ is the fluctuating component of the plasma disk structure that can be represented as $\hat{\rho}_f = \varepsilon_d^2 \tilde{\rho}_f \cos[2(\varphi - \Omega_{ob} t)]$. Then Eq. (10) becomes

$$-z \left\{ \tilde{\rho}_f + \rho_h \right\} \simeq H_h^2 \frac{d}{dz} \tilde{\rho}_f \quad (11)$$

that has the solution

$$\tilde{\rho}_f \simeq -\frac{1}{2} \frac{z^2}{H_h^2} \rho_h^0 \exp\left(-\frac{1}{2} \frac{z^2}{H_h^2}\right). \quad (12)$$

Thus we have a complete expression for the unperturbed density distribution that is

$$\rho_p = \rho_h^0 \exp\left(-\frac{1}{2} \frac{z^2}{H_h^2}\right) \left\{ 1 - \varepsilon_d^2 \cos[2(\varphi - \Omega_{ob} t)] \exp\left(-\frac{1}{2} \frac{z^2}{H_h^2}\right) \right\} \quad (13)$$

and we observe that the fluctuating component of the density profile is double-peaked in z . Clearly, in this case the total confining electric field is not electrostatic.

5. Master Equation

The Master Equation [5] is obtained from the total momentum conservation equation after applying to it the $\mathbf{e}_\varphi \cdot \nabla \times$ operator. In particular $\mathbf{e}_\varphi \cdot \nabla \times \hat{\mathbf{C}} = -\partial \hat{C}_z / \partial R + \partial \hat{C}_R / \partial z$ and, for

$$\hat{\mathbf{C}} = \tilde{\mathbf{C}}(z) \cos[2(\varphi - \omega t)] \cos[k_R (R - R_0)], \quad (5-1)$$

we have $\mathbf{e}_\varphi \cdot \nabla \times \hat{\mathbf{C}} \simeq -\left\{ \sin[k_R (R - R_0)] \hat{C}_z + \cos[k_R (R - R_0)] \partial \hat{C}_z / \partial R \right\} \cos[2(\varphi - \omega t)]$

A microscopic model for uphill diffusion

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**Fisica del Plasma e Matematica Applicata
Il ruolo della matematica nello sviluppo della fisica del plasma**

Consiglio Nazionale delle Ricerche

Roma, 6 Giugno, 2018

Joint work with:

- Errico Presutti (Gran Sasso Science Institute, L'Aquila)
- Anna De Masi (Università dell'Aquila)
- Claudio Giberti (Università degli Studi di Modena e Reggio Emilia)
- Cecilia Vernia (Università degli Studi di Modena e Reggio Emilia)
- Cristian Giardinà (Università degli Studi di Modena e Reggio Emilia)
- Martin Kröger (ETH Zurich)

- Uphill currents in 1D particle system with Kac potential
- Uphill currents in 2D Ising model out of equilibrium

Stationary uphill currents in presence of phase transitions.

- First order phase transitions at **equilibrium**: spontaneous separation of phases giving rise to a gradient without a current (e.g. Van der Waals phase transition in lattice gases with Kawasaki dynamics and Kac potentials).
- **Nonequilibrium**: The Fourier law states that the heat flux is proportional to minus the gradient of the temperature, analogously the Fick law says that the mass flux is proportional to minus the gradient of the mass density. Both laws state that a gradient gives rise to a current.
- Aim of the talk is to investigate the onset of phase transitions in presence of currents, and check how this fits with the Fourier law.
- Observation: the current has the **wrong** sign when the reservoir densities are metastable: **experimental**, **numerical** and some **theoretical** evidence.

The Darken experiment

In old paper by Darken ¹, the author reports of experiments which show **uphill diffusion** of carbon in a welded specimen where the silicon content is concentrated on the left of the weld.

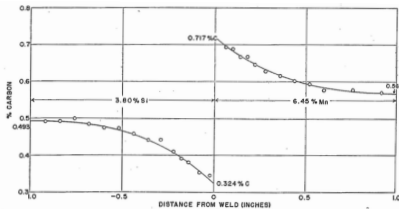


FIG 3—CARBON DISTRIBUTION IN WELDED SPECIMEN NO. 3 AFTER 10 DAYS AT 1050°C.

“For a system of more than two components it is no longer necessary true that a given element tends to diffuse towards a region of lower concentration even within a single phase region.”

[...] Departure from the behavior of an ideal solution may be so great that the concentration gradient and the chemical potential gradient may be of different sign, thus giving rise to “uphill diffusion”. Since, in the discussion of that paper, some skepticism as to the validity of this conclusion was expressed, it seemed desirable to adduce experimental evidence to support it.”

¹L. S. Darken, *Diffusion of carbon in austenite with a discontinuity in composition.*, Trans. AIME **180**, 430 (1949).

A more general class of phenomena

Uphill diffusion appears also in one component systems undergoing a phase transition, e.g. a vapor-liquid transition. **Transient phenomenon** (phase separation) in which mass flows from the lower density to the larger density phase: it has gone **uphill**.

“**Stationary uphill diffusion**” arises when a single component fluid in contact with a left and a right mass reservoir at density $\rho_- < \rho_+$ reaches a stationary state with mass flowing from the reservoir at lower density to the one at larger density.

We have observed this phenomenon in computer simulations.

Theoretical set-up: spin configurations and Kac potentials

Consider a Ising hamiltonian of the form:

$$H_\Lambda(\sigma_\Lambda) = -\frac{1}{2} \sum_{x \neq y \in \Lambda} J(x, y) \sigma_\Lambda(x) \sigma_\Lambda(y) - h \sum_{x \in \Lambda} \sigma_\Lambda(x)$$

- **Classical Ising model:** The only active bonds are those connecting nearest neighbor sites, with ferromagnetic coupling constants all equal to $J > 0$.
- **Mean field models:** Here the coupling constants depend on the region Λ where the system is studied. If Λ has N sites, then $J(x, y) = N^{-1}$, $x, y \in \Lambda$ (range of interaction $\simeq$ size of the system).
- **Kac potentials:** The coupling constants $J_\gamma(x, y)$ depend on a parameter $\gamma > 0$: $J_\gamma(x, y) = \gamma^d J(\gamma x, \gamma y)$, where $J(r, r') = J(r + a, r' + a) \geq 0$, for all $r, r' \in \mathbb{R}^d$; $J(0, r)$ is continuous with compact support and normalized as $\int_{\mathbb{R}^d} J(0, r) dr = 1$.

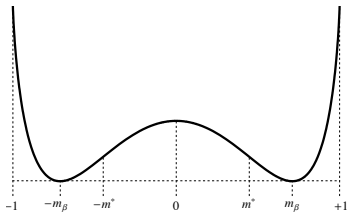
We are interested in the regime in which the lattice distance δ (**microscopic scale**), the interaction length γ^{-1} (**mesoscopic scale**) and the system size N (**macroscopic scale**) satisfy the condition:
 $\delta \ll \gamma^{-1} \ll N$.

This is the ideal regime where the three scales are completely separated, and the **thermodynamics in this regime can be explicitly determined**,

This is the set-up behind some rigorous results concerning the statistical mechanics of Kac potentials, such as the non-local version of the classical scalar **Ginzburg–Landau** functional, namely the “L–P” free energy functional introduced by Lebowitz and Penrose ².

²J. L. Lebowitz, O. Penrose: *Rigorous treatment of the Van der Waals–Maxwell theory of the liquid vapour transition*, J. Math. Phys. **7**, 98 (1966).

The physical system we have in mind is made by a “**channel**” containing a gas of particles and by two density reservoirs which are respectively connected to the right and to the left of the channel and which fix the density of the gas at the endpoints of the channel at values ρ_+ and, respectively, ρ_- .



The temperature is fixed throughout the channel at a value for which there is a phase transition, $\beta > 1$.

The Van der Waals free energy is a double well potential with minima at $\pm m_\beta$, where m_β solves the mean field equation $m_\beta = \tanh\{\beta m_\beta\}$. Moreover, define $m^* : \beta[1 - (m^*)^2] = 1$.

We call the magnetizations $|m| \in [m_\beta, 1]$ **stable**, $|m| \in [m^*, m_\beta)$ **metastable**, $|m| \in [0, m^*)$ **unstable**.

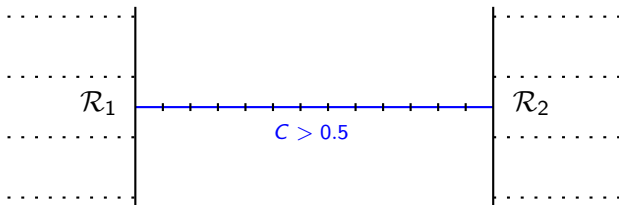
We model the channel as one-dimensional and the gas as a system of particles which interact via a two-body attractive Kac potential, which in the Kac scaling limit gives rise to a van der Waals phase transition. We actually considered two models:

- a lattice gas with **Kawasaki dynamics** and **Kac potential**;
- a stochastic cellular automaton (CA) whose updating rules mimic the Kawasaki dynamics.

The two **microscopic** dynamics give rise to the same **mesoscopic** behavior.

In the sequel we will discuss the CA.

Infinite reservoirs: the OS-CA.



Particles move in a “channel” $\{1, 2, \dots, L\}$, and may also leave from or enter into the channel through L and 1 .

“*Infinite reservoirs*”: no memory of the particles which are absorbed or released.

The CA in the channel is a version of an ASEP. The $d = 1$ SSEP is a system of random walks jumping to the right and left with equal probability (**exclusion principle**).

The **weak asymmetry** that we add is a small bias to jump in the direction where the density is higher.

The phase space is $\mathcal{S} = \{(x, v), x \in \{1, \dots, L\}, v \in \{-1, 1\}\}$, particles configurations are functions $\eta : \mathcal{S} \rightarrow \{0, 1\}$, $\eta(x, v) \in \{0, 1\}$ denotes the occupation variable at (x, v) and v will be interpreted as a **velocity**.

Four more parameters: $\gamma : \gamma^{-1} \in \mathbb{N}$, $C > 0$ and $0 \leq \rho_- < \rho_+ \leq 1$.

Notation:

for $x \in [1, L]$ let $\eta(x) := \eta(x, -1) + \eta(x, 1)$;

for $x \geq 1$, $\eta^{(+)}(x) = \eta(x)$ if $x \in [1, L]$ and $\eta^{(+)}(x) = 2\rho_+$ if $x > L$;

for $x \leq L$, $\eta^{(-)}(x) = \eta(x)$ if $x \in [1, L]$ and $\eta^{(-)}(x) = 2\rho_-$ if $x < 1$.

Finally for $x \in [1, L]$ we call

$$N_{+,x,\gamma} = \sum_{y=x+1}^{x+\gamma^{-1}} \eta^{(+)}(y), \quad N_{-,x,\gamma} = \sum_{y=x-\gamma^{-1}}^{x-1} \eta^{(-)}(y)$$

The unit time step updating is obtained as the result of three successive operations:

- 1 **velocity flip.** At all sites $x \in \{1, \dots, L\}$ where there is only one particle we update its velocity to become $+1$ with probability $\frac{1}{2} + \varepsilon_{x,\gamma}$ and -1 with probability $\frac{1}{2} - \varepsilon_{x,\gamma}$, $\varepsilon_{x,\gamma} = C\gamma^2[N_{+,x,\gamma} - N_{-,x,\gamma}]$. At all other sites the occupation numbers are left unchanged.
- 2 **advection.** After deleting the particles in the channel at $(1, -1)$ and $(L, 1)$ (if present) we let each one of the remaining particles in the channel move by one lattice step in the direction of its velocity.
- 3 **exchanges with the reservoirs.** With probability ρ_+ we put a particle at $(L, -1)$ and with probability $1 - \rho_+$ we leave $(L, -1)$ empty. We do independently the same operations at $(1, 1)$ but with ρ_- instead of ρ_+ .

The simulations exhibit **two totally unexpected phenomena** when the reservoirs densities ρ_- and ρ_+ are such that $\rho_- < \rho_+$, and these values are minus/plus metastable (i.e. metastable and in the two different phases).

In such a case the system seems to reach a **stationary state** such that:

- in a large fraction of the volume the density is metastable;
- the current in the channel becomes positive so that mass goes from the reservoir at lower density to the one with larger density.

Magnetization profiles. We have run several Monte Carlo simulations for different values of the parameters defining the CA, shown are the results in the case $C = 1.25$, $\gamma^{-1} = 30$, $L = 600$ and $\rho_- < \rho_+ = 1 - \rho_-$.

We have computed the local particles density $\rho(x, t)$ by taking the time average

$$\frac{1}{2T} \sum_{s=t}^{t+T-1} \eta_s(x)$$

$\eta_s(x)$ the number of particles at x at time s , $T = L^2$.

Instead of $\rho(x, t)$ we have plotted $m(r, t) = 2\rho(\gamma^{-1}r, t) - 1$, thus the unit space length becomes γ^{-1} (the interaction range) and the density is written in “**magnetization variables**” so that the magnetization at the endpoints is $m_+ = -m_-$.

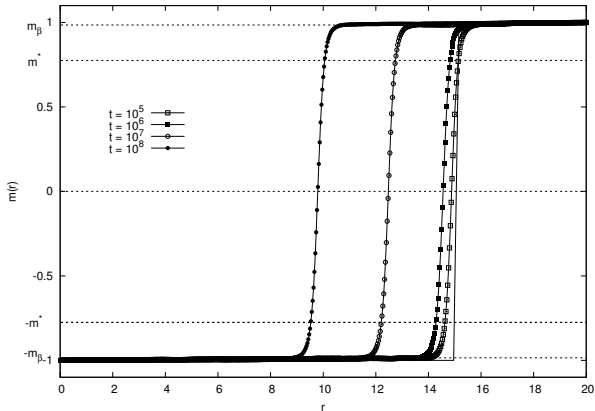


Figure: Magnetization profiles for $C = 1.25$ and $m_+ = 1$ with space in γ^{-1} ($= 30$) units. The parameters m_β and m^* have values $m_\beta = 0.985$ and $m^* = 0.775$. The different curves in the plot correspond to the averaged magnetization computed at different times: $t = 10^5$ (empty squares), $t = 10^6$ (filled squares), $t = 10^7$ (empty circles) and $t = 10^8$ (filled circles). The black thin line denotes the initial configuration, corresponding to a step function centered at $r = 15$.

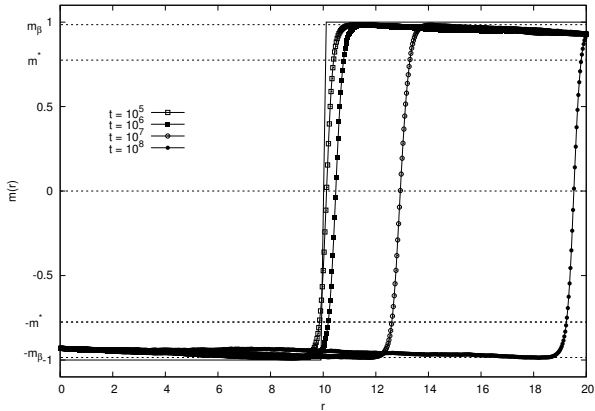


Figure: Magnetization profiles for $C = 1.25$, $m_\beta = 0.985$ and $m^* = 0.775$, and with $m_+ = 0.93$. The initial datum is a step function centered at $r = 10$.

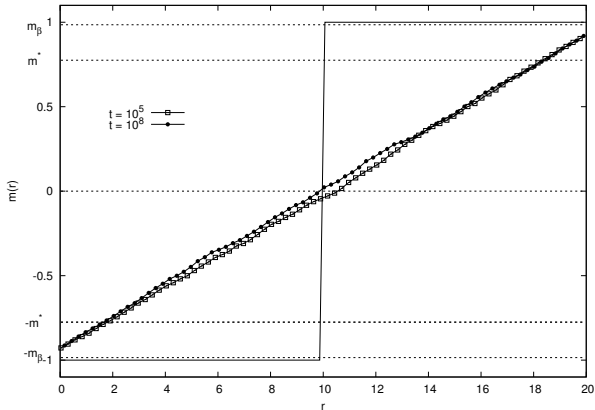


Figure: Magnetization profiles for $C = 0$, $m_\beta = 0.985$ and $m^* = 0.775$, and with $m_+ = 0.93$. The initial datum is a step function centered at $r = 10$. The standard Fourier law is satisfied.

Currents. Besides the magnetization profiles, we have also measured the current by summing (with sign) the total number of particles which in a time interval T at each time step enter into the system from site 0 to site 1 minus those which exit from the channel going from site 1 to site 0.

Since the instantaneous currents are strongly fluctuating, we take averages:

$$j_{\mathcal{R}_1 \rightarrow \text{ch}}^T = \frac{1}{T} \sum_{t=0}^{T-1} j_{\mathcal{R}_1 \rightarrow \text{ch}}(t)$$

Stationarity is reached as $T \rightarrow \infty$, existence of the limit should follow (almost everywhere) from the Birkhoff theorem.

Yet, in the simulations the value of T is chosen empirically in such a way that $j_{\text{ch} \rightarrow \mathcal{R}_2}^T \approx j_{\mathcal{R}_1 \rightarrow \text{ch}}^T$ looks independent of T .

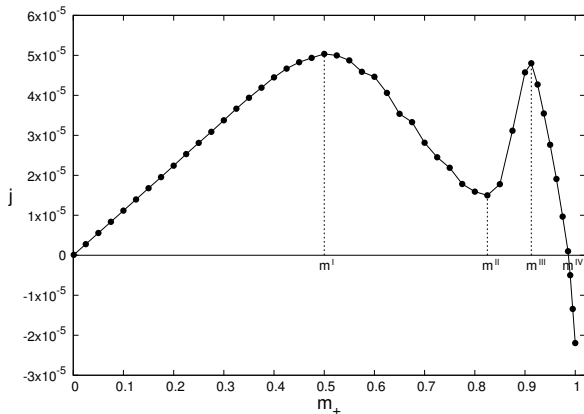


Figure: We plot $j := j_{\text{ch} \rightarrow \mathcal{R}_2}^T$ as a function of m_+ (black dots). The continuous line is $j(m_+)$. Shown are the values $m^I = 0.500$, $m^{II} = 0.825$, $m^{III} = 0.912$, $m^{IV} = 0.985$.

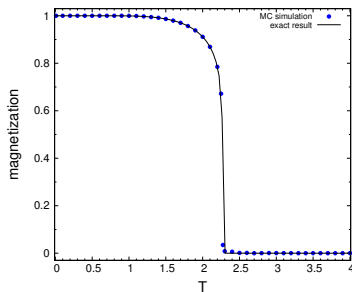
n.n. 2D equilibrium Ising model

- model for ferromagnetism, exactly solvable (Onsager 1944)
- phase transition at the inverse critical temperature:

$$\beta_c = \frac{\ln(1 + \sqrt{2})}{2} \approx 0.440686$$

- spontaneous magnetization:

$$m_\beta = \begin{cases} 0 & \beta \leq \beta_c \\ \left[1 - \frac{1}{\sinh^4(2\beta J)}\right]^{1/8} & \beta > \beta_c \end{cases}$$

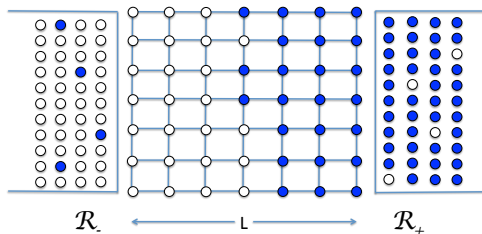


Boundary driven n.n. 2D Ising model

We consider the nearest-neighbor Ising model with ferromagnetic interaction in a square $\Lambda = [0, L]^2 \cap \mathbb{Z}^2$ with Kawasaki dynamics at $\beta > \beta_c$.

Periodic boundary conditions in the vertical direction.

Independent spin flips on the first and last columns that force a magnetization $m_+ > 0$ on the right column and $m_- = -m_+$ on the left column (which simulate two reservoirs).



Spin variable: $\sigma(i) \in \{-1, +1\}$, with $i \in \Lambda$.

Hamiltonian:

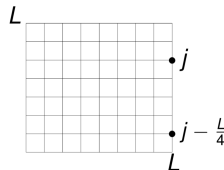
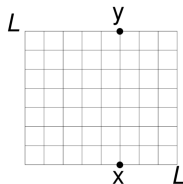
$$H_{\Lambda}(\sigma) = -\frac{1}{2} \sum_{\substack{i,j \in \Lambda \\ |i-j|=1}} \sigma_i \sigma_j + H_{b.c.}(\sigma)$$

- vertical: periodic b.c.

$$\sigma(x, L+1) = \sigma(x, 1)$$

- horizontal: “L/4 b.c.”

$$\begin{aligned} H_{b.c.}(\sigma) &= -\frac{1}{2} \sum_{y=1}^L \sigma_{1,y} \sigma_{1,y-L/4} + \\ &\quad - \frac{1}{2} \sum_{y=1}^L \sigma_{L,y} \sigma_{L,y-L/4} \end{aligned}$$



Continuous time Markov process with rates:

- bulk (**Kawasaki dynamics**):

$$c(i, j) = \begin{cases} 1 & \text{if } \Delta H = H(\sigma^{ij}) - H(\sigma) \leq 0 \\ e^{-\beta \Delta H} & \text{otherwise} \end{cases}$$

- boundaries (**single spin flips**):

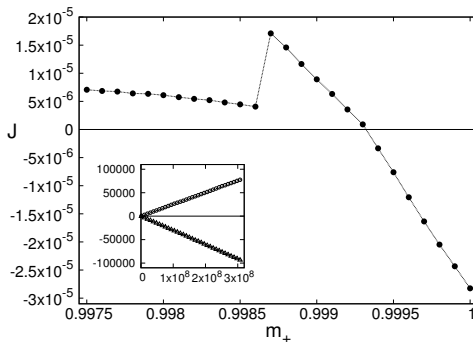
$$c_{-}(i) = \frac{1 - \sigma_i m_{-}}{2} \quad \text{if } i = (1, y)$$

$$c_{+}(i) = \frac{1 - \sigma_i m_{+}}{2} \quad \text{if } i = (L, y)$$

We have done computer simulations using both the classical **Metropolis Monte Carlo** method and the **Kinetic Monte Carlo** method.

While the first algorithm is better suited to describe stationary states the second, which mimics a continuous time dynamics, is useful in the description of transient regimes. The two dynamics have the same invariant measure.

As m_+ decreases from $m_+ = 1$ the current is first **negative** and past a critical value it becomes **positive**.



At $\beta = 1$ we found $m_{crit} \approx 0.99930$ such that:

- for $m_+ > m_{crit}$ it holds $J < 0$, **downhill** (Fick's law);
- for $m_+ < m_{crit}$ it holds $J > 0$, **uphill**.

Note that m_{crit} is very close to $m_\beta = 9.99927$ (Onsager).

Remark: Our results imply no violation of the thermodynamic principles. Indeed, our system (composed of a channel and left/right reservoirs) is not an isolated system. On the contrary, the single spin flip dynamics at the boundaries is such that energy is systematically pumped into the channel.

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- M.C., A. De Masi, E. Presutti, *Latent Heat and the Fourier law*, Phys. Lett. A **380**, 1710 (2016);
- M.C., A. De Masi, E. Presutti, *Particle models with self sustained current*, J. Stat. Phys. **167**, 1081 (2017).
- M.C., A. De Masi, E. Presutti, *Microscopic models for uphill diffusion*, J. Phys. A: Math. Theor. **50**, 435002 (2017).
- E. N. M. Cirillo, M.C., *Locally induced stationary uphill currents*, Phys. Rev. E **96**, 052137 (2017).
- M.C., C. Giardinà, C. Giberti, C. Vernia, *Non-equilibrium 2D Ising model with stationary uphill diffusion*, Phys. Rev. E **97**, 030103(R) (2018).

Thank you for your attention!

Spectral analysis for isogeometric discretizations of curl-div problems

Carla Manni

University of Rome “Tor Vergata”

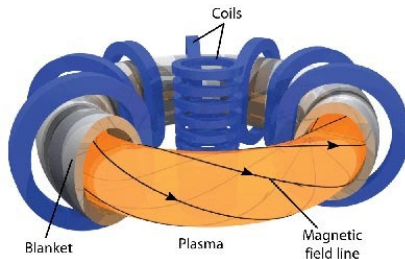
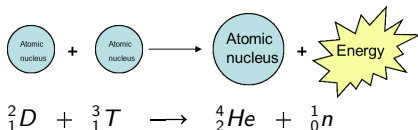
Joint work with

*Mariarosa Mazza · Ahmed Ratnani · Stefano Serra-Capizzano · Hendrik
Speleers*

Fisica del Plasma e Matematica Applicata
Rome, June 6, 2018

Let me tell you about a dream ...

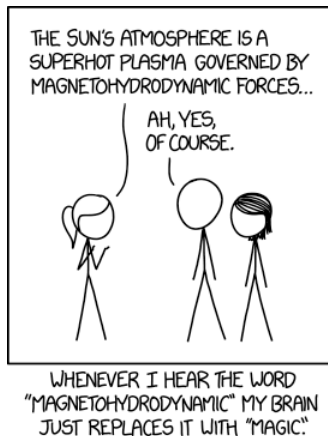
- **Fusion energy** attempts to harness fusion reactions as a source of large scale sustainable energy
- The fuel is heated to temperatures > 150 millions $^{\circ}\text{C}$



- **Tokamak** is toroidal device used for producing energy through controlled thermonuclear fusion
- The hot plasma is confined in the core region using a magnetic field
- **ITER** aims to be the world's largest tokamak able to produce more energy than is required to initiate and sustain a fusion reaction

MagnetoHydroDynamics (MHD)

MHD is combination of Navier-Stokes and Maxwell equations to model plasma



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Involved operators in a 2D/3D vector setting ($d = 2, 3$):

- **curl-curl**: $(\nabla \times \mathbf{u}, \nabla \times \mathbf{v})$, with $\mathbf{u}, \mathbf{v} : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ in

$$H(\text{curl}, \Omega) := \{\mathbf{u} \in (L^2(\Omega))^d \text{ s.t. } \nabla \times \mathbf{u} \in (L^2(\Omega))^d\}$$

- **div-div**: $(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})$ with $\mathbf{u}, \mathbf{v} : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ in

$$H(\text{div}, \Omega) := \{\mathbf{u} \in (L^2(\Omega))^d \text{ s.t. } \nabla \cdot \mathbf{u} \in L^2(\Omega)\}$$

- **zero order**: $(\mathbf{u}, \mathbf{v})$ with $\mathbf{u}, \mathbf{v} : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ either in $H(\text{curl}, \Omega)$ or $H(\text{div}, \Omega)$
- **combinations** of the previous ones

Alfvén operator

Among the different operators, one encounters the **Alfvén-like operator**

$$\mathcal{L}_A \mathbf{u} := \nu \mathbf{u} - \beta \lambda \nabla (\nabla \cdot \mathbf{u}) - \lambda (\mathbf{b}_0 \times (\nabla \times \nabla \times (\mathbf{b}_0 \times \mathbf{u})))$$

where

- $\nu > 0, \beta \in (10^{-4}, 10^{-1}]$
- λ is the numerical Alfvén length
- $\mathbf{b}_0 := \frac{\mathbf{B}_0}{\|\mathbf{B}_0\|}$, with $\mathbf{B}_0$ the magnetic field

Motivation

The operator $\mathcal{L}_A$ needs to be inverted in an optimal way, using an algorithm with a high scalability property over a computational domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$



We need to understand the competition between the curl and div terms

Curl-div as a stabilization for curl-curl

Our problem

Find $\mathbf{u} \in (H_0^1(\Omega))^d$ such that

$$\alpha(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + \beta(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in (H_0^1(\Omega))^d$$

with

- parameters: $\alpha \sim 1$ and $\beta \in (10^{-4}, 10^{-1}]$
- domain: $\Omega = (0, 1)^d$, $d = 2, 3$
- discretisation: isogeometric analysis with B-splines

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Our aim

Make a fast solver for the corresponding linear systems (based on the symbol)

Symbol in a specific case: Toeplitz matrices

Let $\{T_n(f)\}_n$ be **Toeplitz sequence generated by f** with

$$T_n(f) := \begin{bmatrix} f_0 & f_{-1} & \cdots & f_{-(n-1)} \\ f_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{-1} \\ f_{n-1} & \cdots & f_1 & f_0 \end{bmatrix}, \text{ and } \begin{aligned} f &\in L^1[-\pi, \pi] \\ f_j &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ij\theta} d\theta, \quad j \in \mathbb{Z} \end{aligned}$$

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An equispaced sampling of f over $[-\pi, \pi]$ gives an approximation of the spectrum of $T_n(f)$ for large $n \rightarrow$ **f is called the symbol of $\{T_n(f)\}_n$**

$$f \text{ \textbf{real-valued}: } \lambda_j(T_n(f)) \approx f\left(-\pi + \frac{2\pi j}{n-1}\right), \quad j = 0, \dots, n-1$$

$$f \text{ \textbf{complex-valued}: } \sigma_j(T_n(f)) \approx \left| f\left(-\pi + \frac{2\pi j}{n-1}\right) \right|, \quad j = 0, \dots, n-1$$

Informal definition of symbol

Assume

- $\{A_n\}_n$ is matrix-sequence with $\dim(A_n) = d_n \rightarrow \infty$
- $f : D \subset \mathbb{R}^d \rightarrow \mathbb{C}$ measurable, $0 < \text{measure}(D) < \infty$

f is the **spectral symbol** of $\{A_n\}_n$ in the sense that the eigenvalues of A_n are approximately uniform sampling of f over D for large n

- $\{A_n\}_n$ has a **spectral distribution** described by f
- Notation: $\{A_n\}_n \sim_\lambda (f, D)$

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Remark: this definition can also be given in the singular value sense (replacing $f \rightarrow |f|$). Notation: $\{A_n\}_n \sim_\sigma (f, D)$

Remark: low-rank and small-norm perturbations do not affect the symbol

Remark: *-algebra available for computation of symbols (GLT theory)

Outline

A guide to the talk... the importance of the symbol

For our curl-div problem:

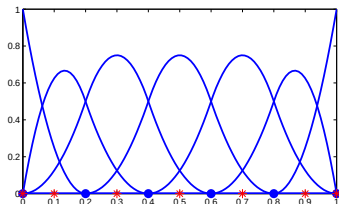
- 1 Construction of the matrix and computation of symbol
- 2 Analysis of the spectral properties of the matrix
- 3 Design of fast (multigrid) solvers based on the symbol

Construction of curl-div matrix: isogeometric approach

- 1D: spline approximation space $\mathbb{S}_n^p$

- domain $\Omega = (0, 1)$
- degree p , mesh size $h = 1/n$
- B-spline basis

$$\{N_i^p : i = 2, \dots, n + p - 1\}$$



$p = 2$, open knot vector

- 2D: tensor-product $\mathbb{S}_{n_1}^p \otimes \mathbb{S}_{n_2}^p$
- 3D: tensor-product $\mathbb{S}_{n_1}^p \otimes \mathbb{S}_{n_2}^p \otimes \mathbb{S}_{n_3}^p$

higher accuracy per DoF w.r.t. FEM

Construction of curl-div matrix: isogeometric approach

Coefficient matrix is a $d \times d$ block matrix

For example, in 2D:

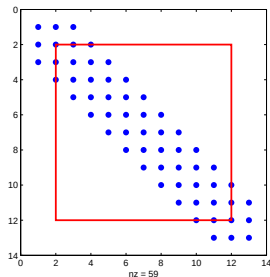
$$\mathcal{A}_n^{\alpha, \beta} = \alpha \begin{bmatrix} M_{n_1}^p \otimes S_{n_2}^p & -(A_{n_1}^p)^T \otimes A_{n_2}^p \\ -A_{n_1}^p \otimes (A_{n_2}^p)^T & S_{n_1}^p \otimes M_{n_2}^p \end{bmatrix} + \beta \begin{bmatrix} S_{n_1}^p \otimes M_{n_2}^p & A_{n_1}^p \otimes (A_{n_2}^p)^T \\ (A_{n_1}^p)^T \otimes A_{n_2}^p & M_{n_1}^p \otimes S_{n_2}^p \end{bmatrix}$$

with $\mathbf{n} = (n_1, n_2)$,

- $M_n^p = \left[\int_0^1 N_{i+1}^p(t) N_{j+1}^p(t) dt \right]_{i,j}$ 1D mass matrix
- $A_n^p = \left[\int_0^1 N_{i+1}^p(t) (N_{j+1}^p(t))' dt \right]_{i,j}$ 1D advection matrix
- $S_n^p = \left[\int_0^1 (N_{i+1}^p(t))' (N_{j+1}^p(t))' dt \right]_{i,j}$ 1D stiffness matrix
- when $\alpha = \beta = 1$ we get vector Laplacian matrix

Symbol of 1D matrices

Due to uniform mesh, the **central part** of the 1D matrices has **Toeplitz-structure**



$$p = 2$$

Symbol of 1D matrices

Due to uniform mesh, the **central part** of the 1D matrices has **Toeplitz-structure**

- $\{nM_n^p\}_n \sim_\lambda (m_p, [-\pi, \pi])$

$$m_p(\theta) = \phi_{2p+1}(p+1) + 2 \sum_{k=1}^p \phi_{2p+1}(p+1-k) \cos(k\theta)$$

- $\{-iA_n^p\}_n \sim_\lambda (a_p, [-\pi, \pi])$

$$a_p(\theta) = -2 \sum_{k=1}^p \phi'_{2p+1}(p+1-k) \sin(k\theta)$$

- $\{\frac{1}{n}S_n^p\}_n \sim_\lambda (s_p, [-\pi, \pi])$

$$s_p(\theta) = -\phi''_{2p+1}(p+1) - 2 \sum_{k=1}^p \phi''_{2p+1}(p+1-k) \cos(k\theta)$$

with **cardinal B-spline** ϕ_{2p+1} on knots $\{0, 1, \dots, 2p+2\}$

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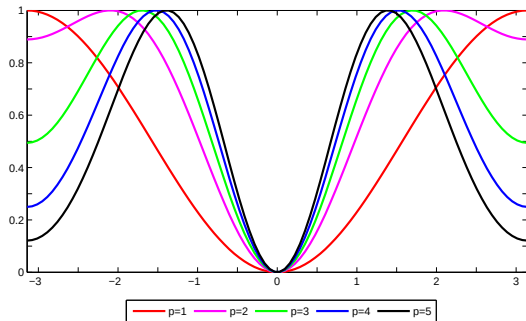
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$$\begin{aligned} s_p(\theta) &= -\phi''_{2p+1}(p+1) - 2 \sum_{k=1}^p \phi''_{2p+1}(p+1-k) \cos(k\theta) \\ &= (2 - 2\cos(\theta))m_{p-1}(\theta) \end{aligned}$$

with **cardinal B-spline** ϕ_{2p+1} on knots $\{0, 1, \dots, 2p+2\}$

Symbol of 1D matrices

Properties of the symbol s_p



graph of normalized s_p

Symbol of 1D matrices

Properties of the symbol s_p

- ill-conditioning at the **low frequencies** ($\theta = 0$)
canonical: similar to FD/FE discretizations
- ill-conditioning at the **high frequencies for large p** ($\theta = \pi$)
non-canonical

Symbol of 1D matrices

Properties of the symbol s_p

- ill-conditioning at the **low frequencies** ($\theta = 0$)
canonical: similar to FD/FE discretizations
- ill-conditioning at the **high frequencies for large p** ($\theta = \pi$)
non-canonical



classical multigrid methods
present problems for large p
e.g., based on classical smoother

Symbol of curl-div matrix

Coefficient matrix is permutation of d -level block Toeplitz + low-rank

For example, in 2D:

$$\mathcal{A}_n^{\alpha,\beta} = \alpha \begin{bmatrix} M_n^p \otimes S_n^p & -(A_n^p)^T \otimes A_n^p \\ -A_n^p \otimes (A_n^p)^T & S_n^p \otimes M_n^p \end{bmatrix} + \beta \begin{bmatrix} S_n^p \otimes M_n^p & A_n^p \otimes (A_n^p)^T \\ (A_n^p)^T \otimes A_n^p & M_n^p \otimes S_n^p \end{bmatrix}$$

has symbol

$$f^{p,\alpha,\beta} = \alpha \begin{bmatrix} m_p \otimes s_p & -a_p \otimes a_p \\ -a_p \otimes a_p & s_p \otimes m_p \end{bmatrix} + \beta \begin{bmatrix} s_p \otimes m_p & a_p \otimes a_p \\ a_p \otimes a_p & m_p \otimes s_p \end{bmatrix}$$

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$$\mathcal{A}_n^{\alpha,\beta} = \alpha \begin{bmatrix} M_n^p \otimes S_n^p & -(A_n^p)^T \otimes A_n^p \\ -A_n^p \otimes (A_n^p)^T & S_n^p \otimes M_n^p \end{bmatrix} + \beta \begin{bmatrix} S_n^p \otimes M_n^p & A_n^p \otimes (A_n^p)^T \\ (A_n^p)^T \otimes A_n^p & M_n^p \otimes S_n^p \end{bmatrix}$$

has symbol

$$f^{p,\alpha,\beta} = \alpha \begin{bmatrix} m_p \otimes s_p & -a_p \otimes a_p \\ -a_p \otimes a_p & s_p \otimes m_p \end{bmatrix} + \beta \begin{bmatrix} s_p \otimes m_p & a_p \otimes a_p \\ a_p \otimes a_p & m_p \otimes s_p \end{bmatrix}$$

Eigenvalues of $\mathcal{A}_n^{\alpha,\beta}$ can be approximated by uniform sampling of

$$\lambda_1(f^{p,\alpha,\beta}), \quad \lambda_2(f^{p,\alpha,\beta})$$

over $[0, \pi]^2$

Symbol of curl-div matrix

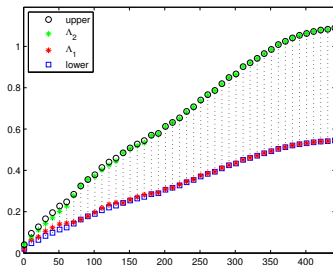
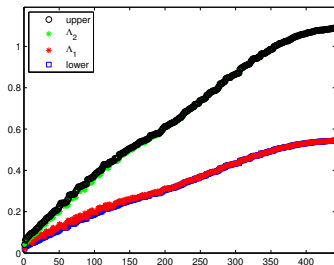
2D Laplacian symbol is $L_p = m_p \otimes s_p + s_p \otimes m_p$

$$0 \leq \min(\alpha, \beta) L_p \leq \lambda_1(f^{p,\alpha,\beta}) \leq \lambda_2(f^{\alpha,\beta}) \leq \max(\alpha, \beta) L_p$$

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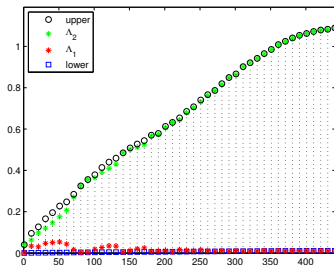
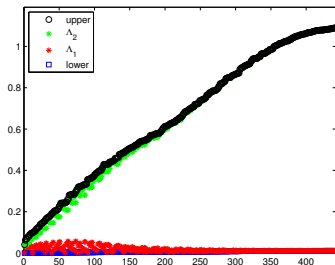


$$n = 20, p = 3, \alpha = 1, \beta = 0.5$$

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$$n = 20, p = 3, \alpha = 1, \beta = 0.01$$

Symbol of curl-div matrix

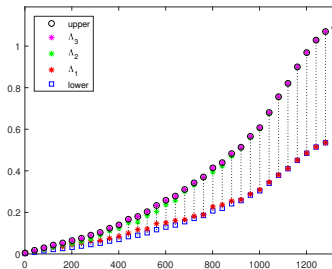
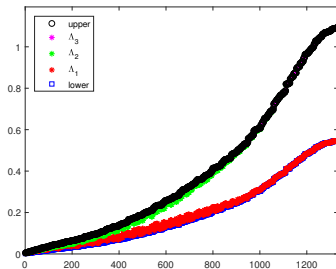
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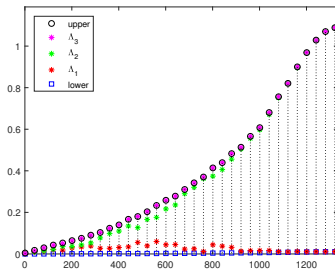
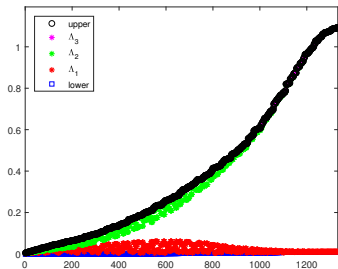


$$n = 10, p = 3, \alpha = 1, \beta = 0.5$$

Symbol of curl-div matrix

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$$n = 10, p = 3, \alpha = 1, \beta = 0.01$$

Symbol of curl-div matrix

Properties of the symbol $f^{p,\alpha,\beta}$

Different sources of ill-conditioning:

- $\lambda_i(f^{p,\alpha,\beta})$ behaves like the Laplacian symbol (recall properties s_p)
 - (1) zero at origin $\Rightarrow$ canonical ill-conditioning in the **low frequencies**
 - (2) close to zero at π -edges for large $p \Rightarrow$ ill-conditioning in the **high frequencies**

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 - (1) zero at origin $\Rightarrow$ canonical ill-conditioning in the **low frequencies**
 - (2) close to zero at π -edges for large $p \Rightarrow$ ill-conditioning in the **high frequencies**
- $\lambda_1(f^{p,\alpha,\beta})$ behaves like β for very small β
 - (3) $1/d$ of spectrum close to zero $\Rightarrow$ huge ill-conditioning in the **low frequencies**

Fast solvers

Classical multigrid methods

- take care of the zero at origin
⇒ optimal, i.e., convergence rate independent of n
- do not take care of other zeros, e.g., at π -edges for large p
⇒ a very bad convergence rate (close to 1)

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Multi-iterative methods: combine the best of everything

- keep classical transfer (e.g., interpolation) for dealing with zero at origin
- replace classical smoother (e.g., Gauss-Seidel) with another smoother that takes care of the other (numerical) zeros of the symbol
⇒ only necessary at finest level

Fast solvers

Multi-iterative methods

- Take PCG or PGMRES as smoother, with a preconditioner having symbol g_p which erases the numerical zeros of our symbol f_p
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- Examples:
 - 1D Laplacian: we have $f_p = s_p = (2 - 2\cos)m_{p-1}$, so $g_p = m_{p-1}$
 - multi-D Laplacian: use a preconditioner with symbol

$$g_p = m_{p-1} \otimes m_{p-1} \otimes \cdots \otimes m_{p-1}$$

possible choice: Toeplitz matrix generated by g_p

$$T(g_p) = T(m_{p-1}) \otimes T(m_{p-1}) \otimes \cdots \otimes T(m_{p-1})$$

Multi-iterative method (MIM)

MIM for general curl-div problem

Thanks to the bounds

$$0 \leq \min(\alpha, \beta) L_p \leq \lambda_i(\mathbf{f}^{p, \alpha, \beta}) \leq \max(\alpha, \beta) L_p$$

we can apply the same **MIM** to matrix $\mathcal{A}_n^{\alpha, \beta}$ for general α and β

- we expect **robustness with respect to p**
- we expect **optimality with respect to the matrix-size**
- a first attempt to guarantee robustness with respect to α, β is to use the MIM as preconditioner for the CG

Numerical results

2D vector Laplacian example: $\alpha = 1, \beta = 1$

$p = 1$				$p = 2$				$p = 3$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
16	3	6	25	15	2	4	14	14	2	4	27
32	3	6	50	31	3	5	30	30	3	5	37
64	3	6	101	63	3	6	61	62	3	6	70

$p = 4$				$p = 5$				$p = 6$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
13	2	3	50	12	2	4	91	11	3	5	137
29	2	4	57	28	2	4	108	27	2	4	220
61	3	6	73	60	3	6	113	59	3	5	210

P_{MIM} is the preconditioner given by one iteration of MIM applied to matrix $\mathcal{A}_n^{p,\alpha,\beta}$
 tol = 10^{-7} (Matlab implementation)

Numerical results

2D curl-div example: $\alpha = 1$, $\beta = 0.1$

$p = 1$				$p = 2$				$p = 3$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
16	7	21	57	15	6	15	40	14	5	12	48
32	7	21	123	31	6	15	77	30	5	12	73
64	7	21	252	63	6	15	153	62	6	14	150

$p = 4$				$p = 5$				$p = 6$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
13	5	11	82	12	5	10	125	11	6	11	206
29	5	12	113	28	5	11	206	27	5	12	333
61	5	13	167	60	5	13	267	59	6	13	475

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Numerical results

2D curl-div example: $\alpha = 1$, $\beta = 0.01$

$p = 1$				$p = 2$				$p = 3$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
16	18	122	115	15	16	114	91	14	15	94	91
32	19	140	291	31	17	116	207	30	16	94	201
64	21	152	685	63	18	116	427	62	17	102	419

$p = 4$				$p = 5$				$p = 6$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
13	15	88	107	12	15	80	149	11	16	88	223
29	15	87	207	28	16	85	292	27	17	80	436
61	17	96	435	60	17	92	451	59	17	89	680

P_{MIM} is the preconditioner given by one iteration of MIM applied to matrix $\mathcal{A}_n^{p,\alpha,\beta}$
 tol = 10^{-7} (Matlab implementation)

Numerical results

3D curl-div example: $\alpha = 1$, $\beta = 0.01$

$p = 1$				$p = 2$				$p = 3$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
8	12	61	38	7	15	106	46	6	13	88	72
16	17	121	114	15	16	112	103	14	14	87	102
32	20	158	292	31	16	120	205	30	15	94	202

$p = 4$				$p = 5$				$p = 6$			
n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG	n	P_{MIM}	MIM	CG
5	16	72	114	4				3			
13	15	83	142	12	17	88	229	11	19	105	360
29	15	85	214	28	15	81	316	27	17	80	487

P_{MIM} is the preconditioner given by one iteration of MIM applied to matrix $\mathcal{A}_n^{p,\alpha,\beta}$
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Concluding message

The symbol approach in IgA matrix analysis

- It is useful for understanding spectral features and designing fast solvers
- We identify **two steps in the symbol approach**:
 - **derive the symbol** for the PDE discretization matrix and study its properties (using GLT theory)
 - **use the symbol** and its properties to design efficient iterative solvers of Krylov, multigrid, or multi-iterative type

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Ongoing work and future tasks

- Deal with the parameters α, β : the matrix can be highly ill-conditioned, so a **regularization strategy** is recommendable
- Extend this approach to the case with **mapping** (toroidal geometry)
- Spectral analysis and related iterative strategies for the **Alfvén operator**

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Rome – June 6, 2018

Inflexional instability and twist bursts of magnetic fields

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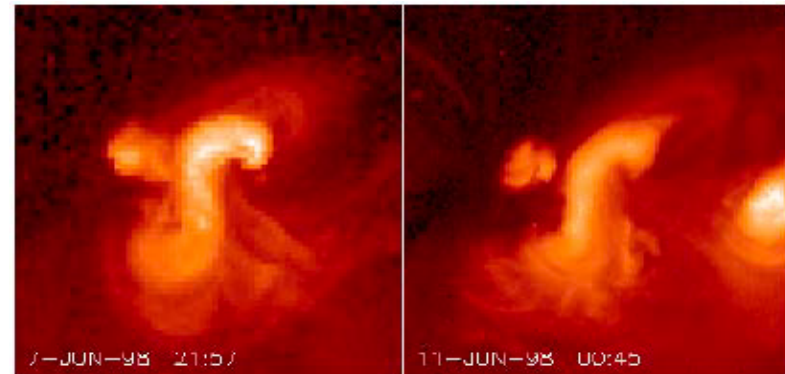
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*Soft X-ray sigmoids in the
Sun's south emisphere*
(Yohkoh 1998)

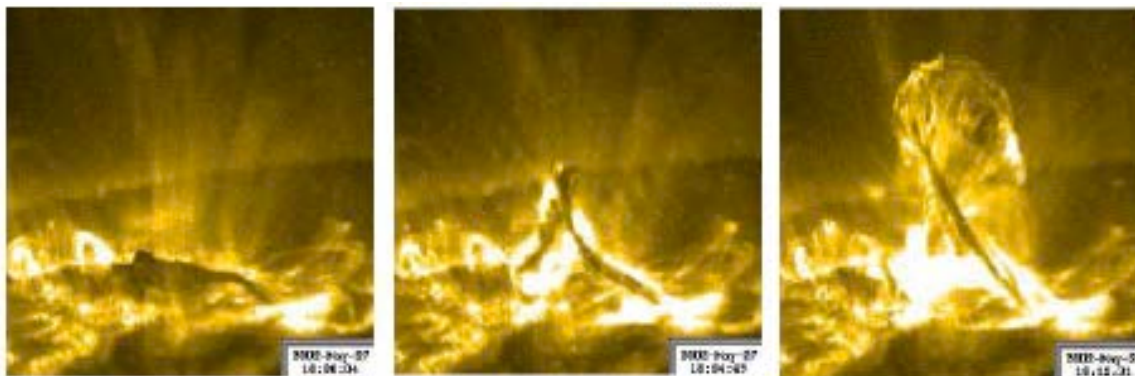
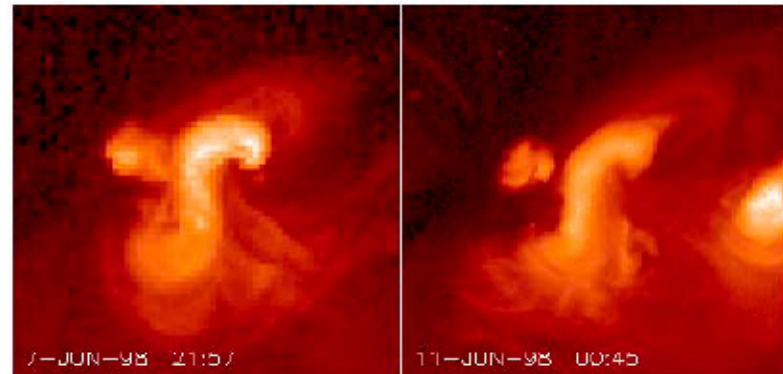


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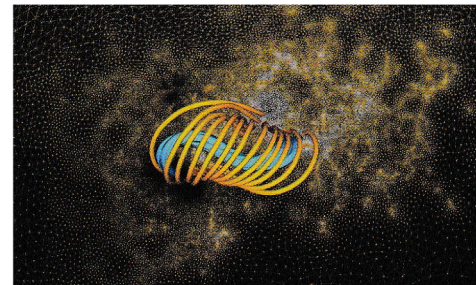
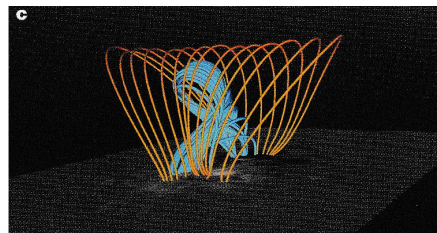
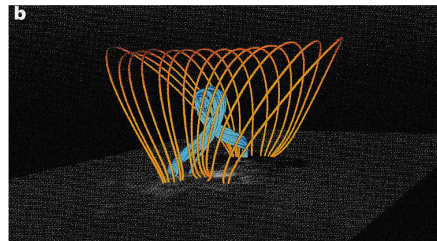
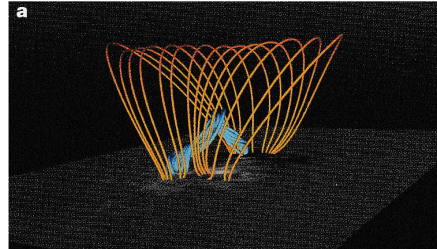
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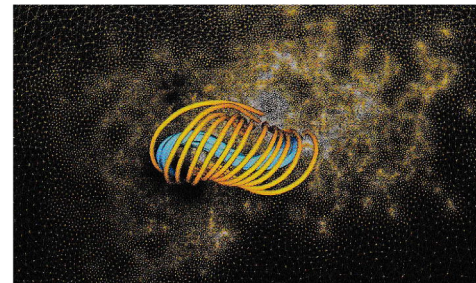
*Thin filament
destabilization
(Trace 2002)*

Instability triggered by twist localization

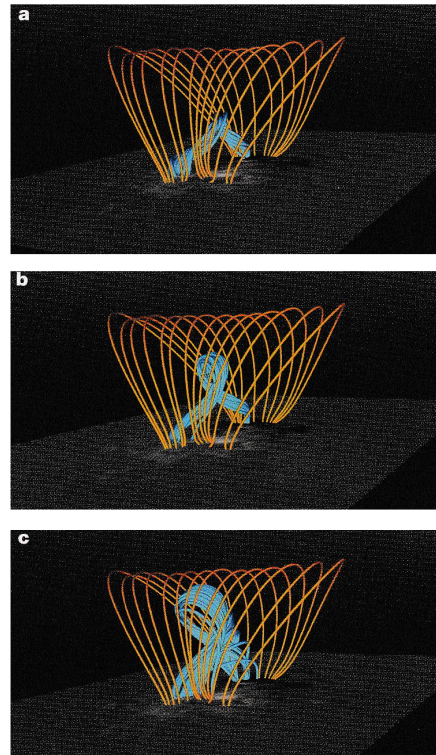


(Amari et al, Nature 2018)

Instability triggered by twist localization



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EVOLUTION AND INFLEXIONAL INSTABILITY OF TWISTED MAGNETIC FLUX TUBES

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(Received 30 December 1996; accepted 28 January 1997)

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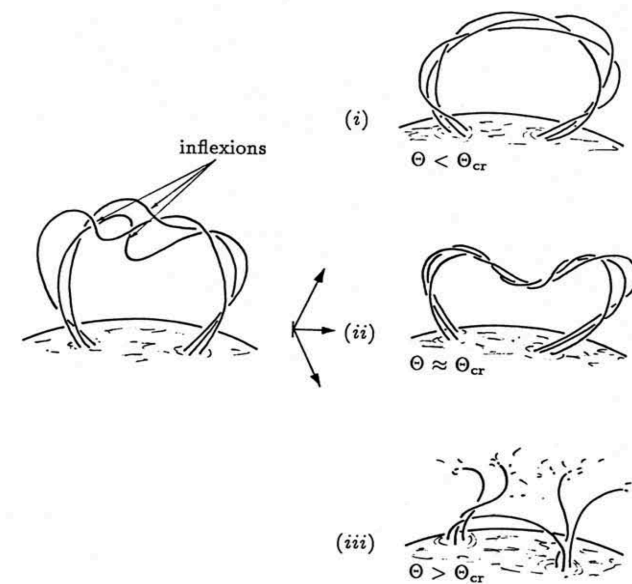


Figure 2. A coronal loop is identified with a bundle of magnetic fields twisted-up by the vortical photospheric motion that during evolution may develop inflexional configurations in disequilibrium. Depending on the amount of twist actually present in the loop, we have: (i) magnetic field lines relaxing into a braid pattern; (ii) flux tube developing a hammock configuration; or (iii) passage through inflexion, triggering kink instability and eventual outbreak of energy.

Solar Physics 172: 241–248, 1997.

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(Ricca Solar Phys 1997)

Magnetic field and helicity

- *Frozen evolution:* $\mathbf{B}(\mathbf{x}, t) \in \left\{ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) \wedge \nabla \cdot \mathbf{B} = 0; L_2 - \text{norm} \right\}$

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In ideal conditions the magnetic helicity H of a single flux tube of flux Φ , is given by

$$H = \int_V \mathbf{A} \cdot \mathbf{B} d^3\mathbf{x} = \Phi^2 SL = \Phi^2 (Wr + Tw) .$$

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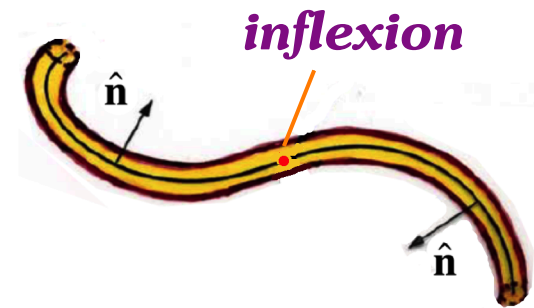


- **Total twist:** $Tw(R) = T(C) + N(R)$ $\left\{ \begin{array}{l} T(C) = \frac{1}{2\pi} \oint_C \tau(s) ds \\ N(R) = [\theta]/2\pi \end{array} \right.$



Passage through an inflexional configuration

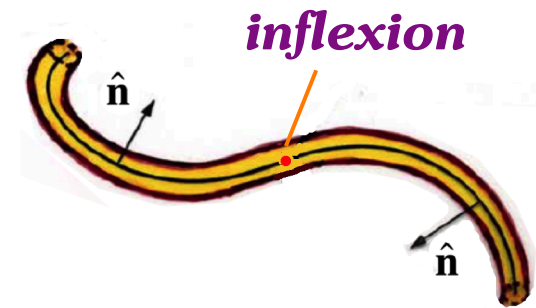
- **Inflexional configuration:** $c = 0$;



Passage through an inflexional configuration

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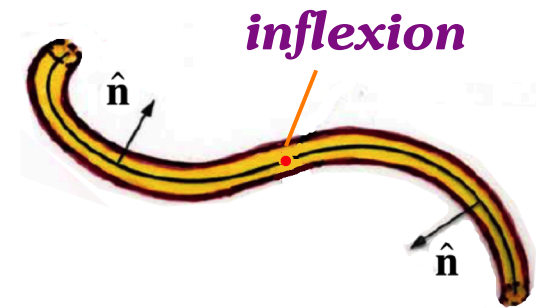


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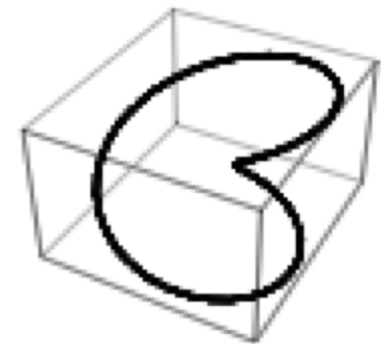
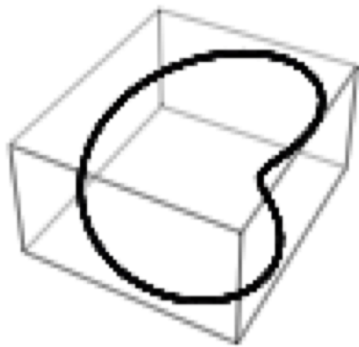
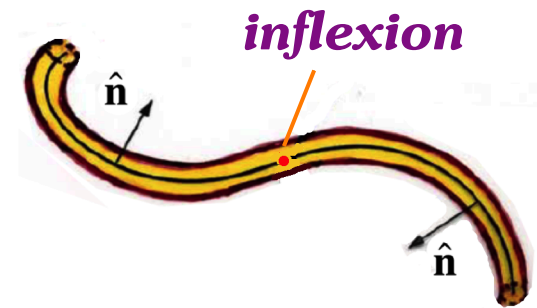


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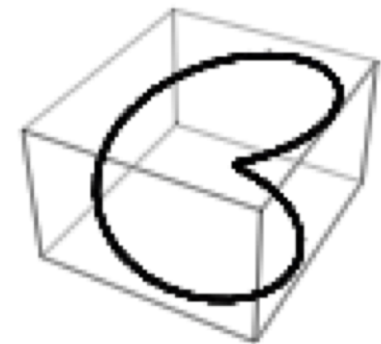
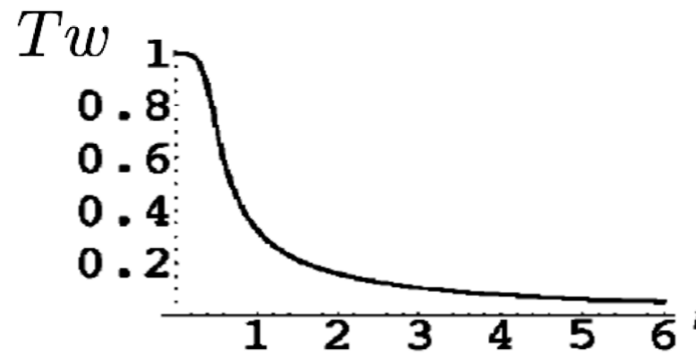
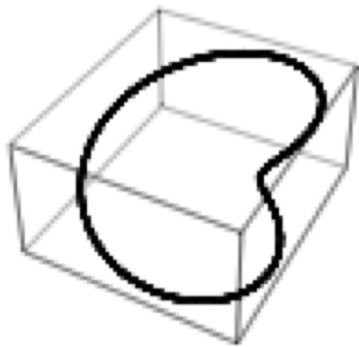
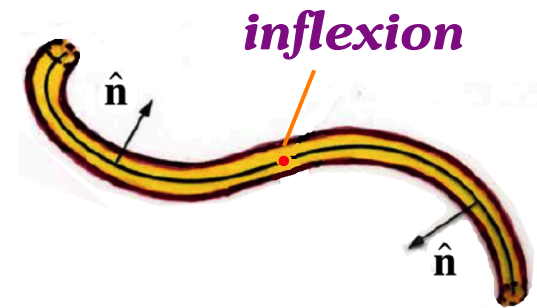


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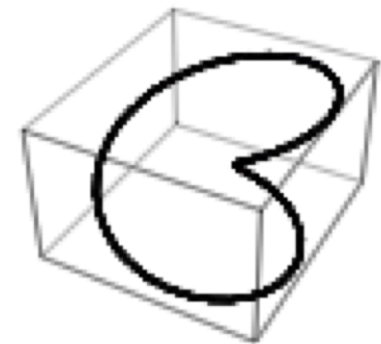
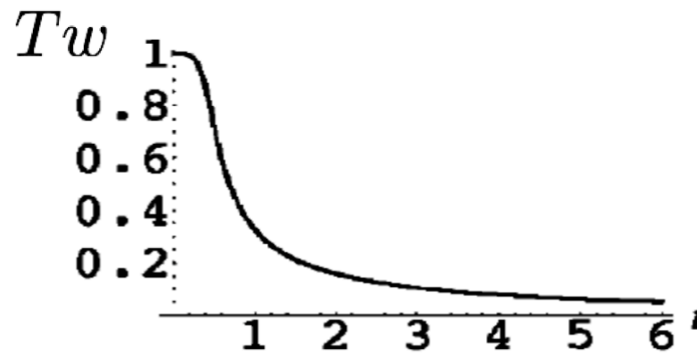
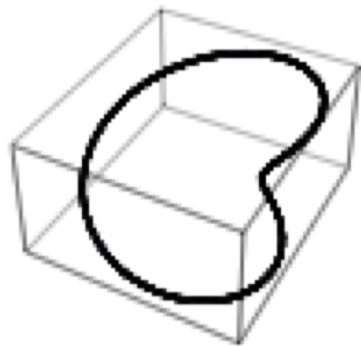
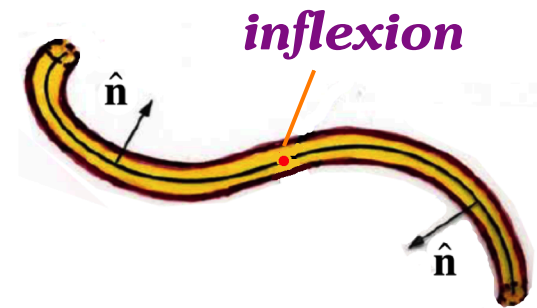


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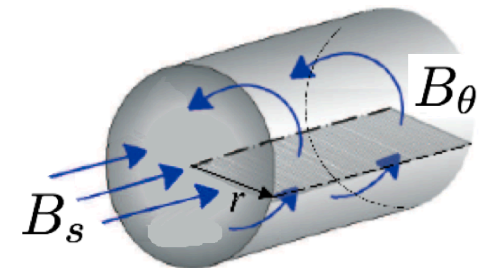
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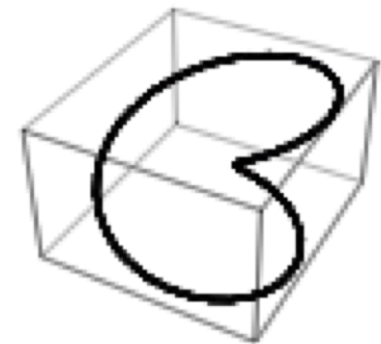
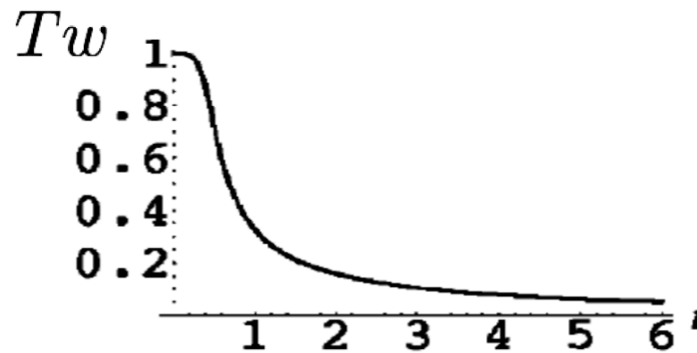
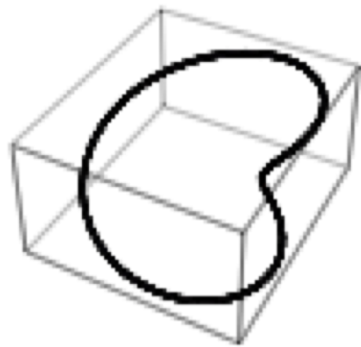
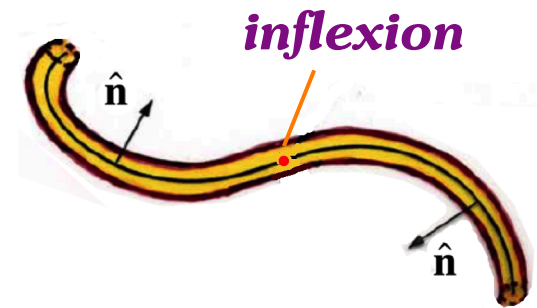


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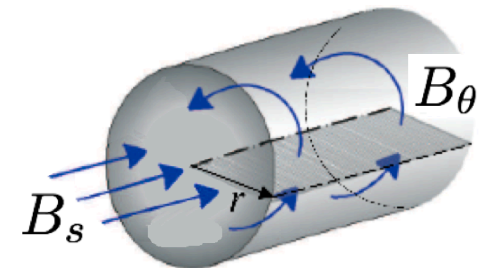
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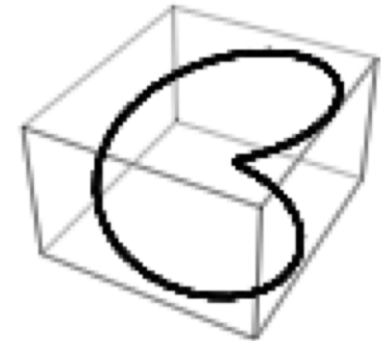
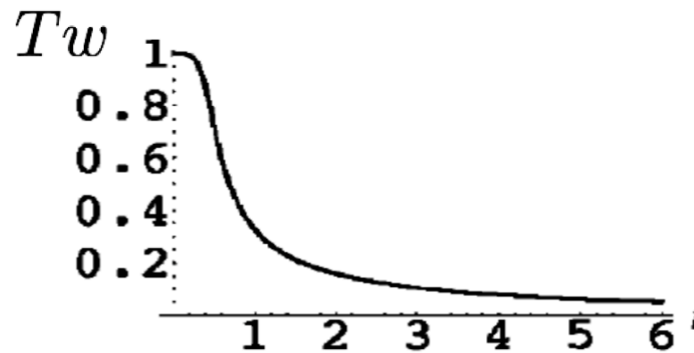
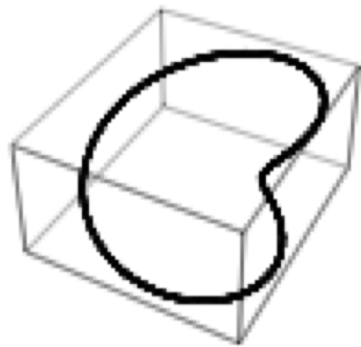
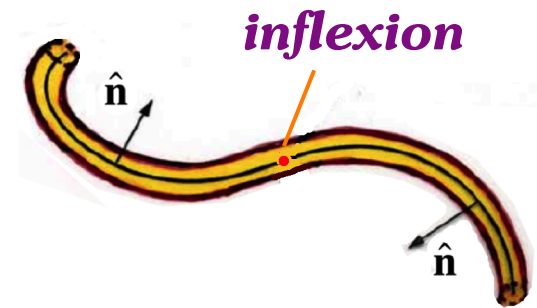


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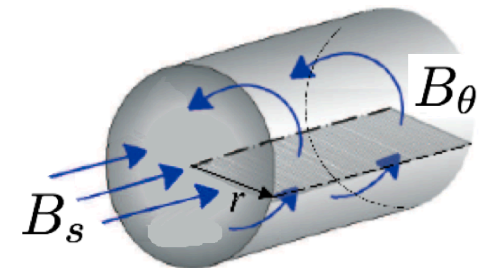
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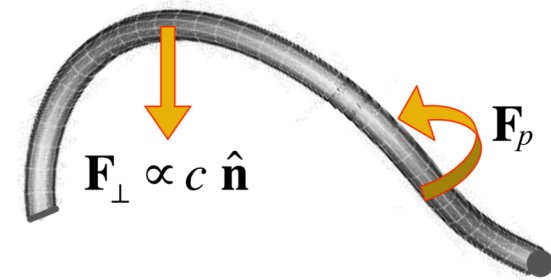
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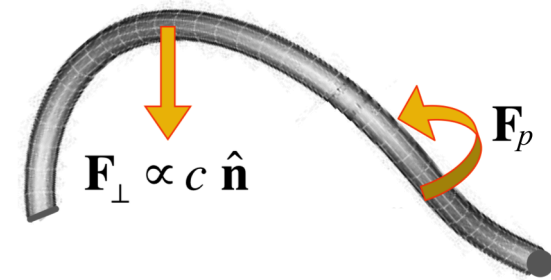
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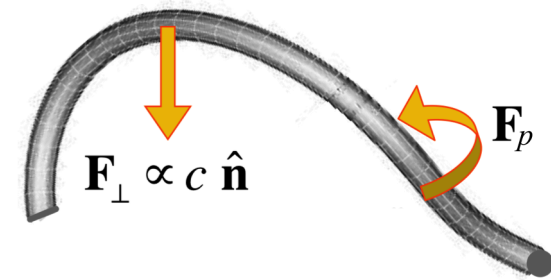
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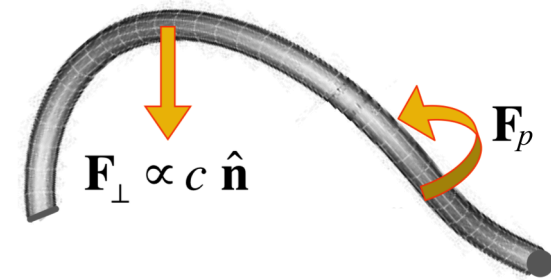
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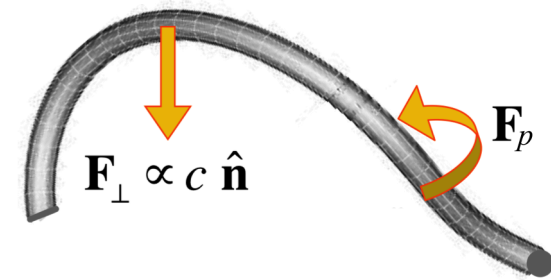
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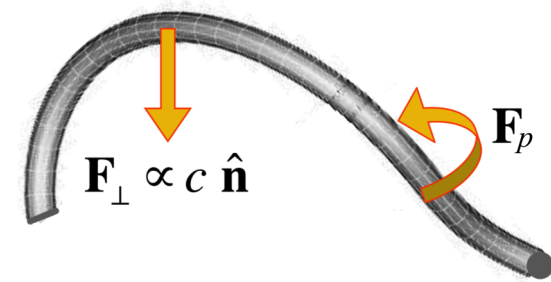
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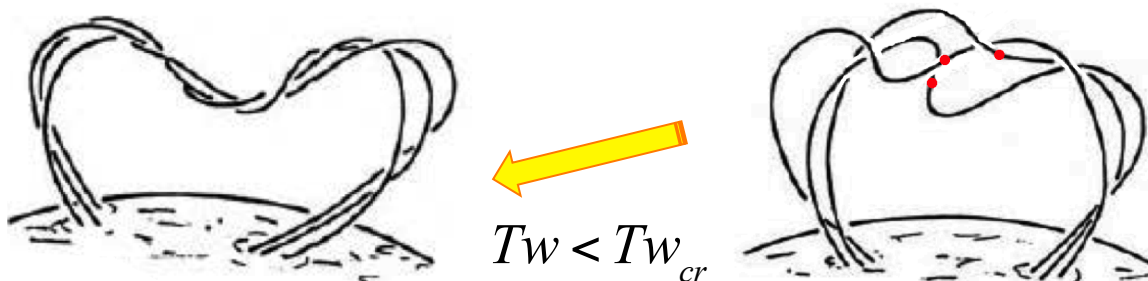
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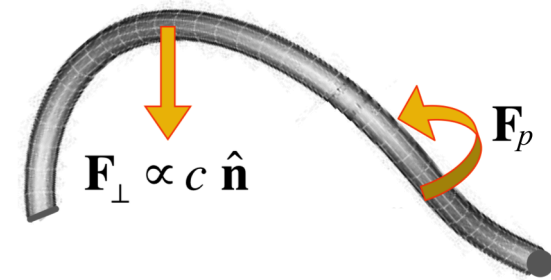
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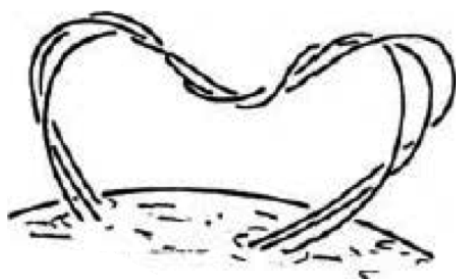
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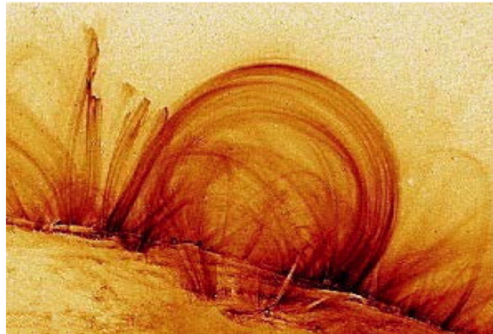


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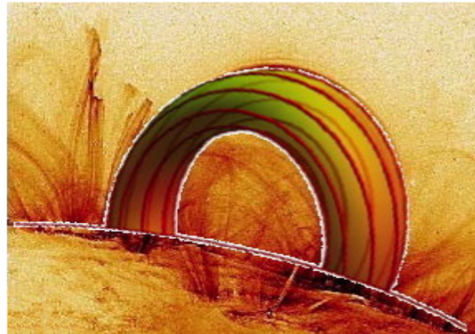
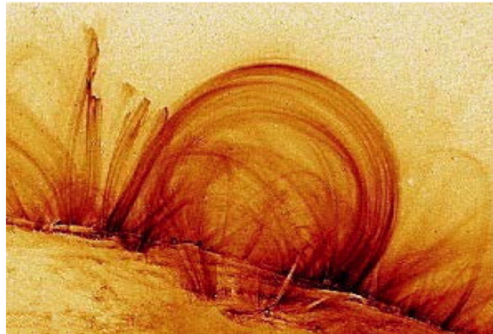
Force-free magnetic fields in toroidal geometry

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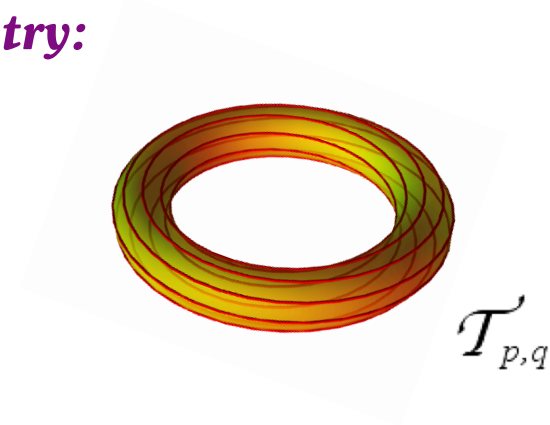
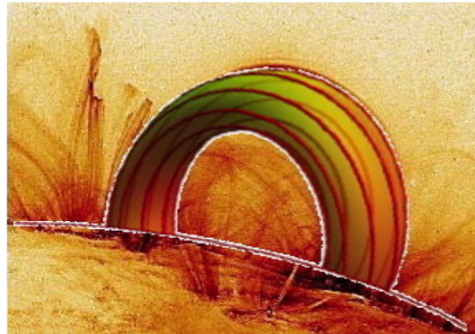
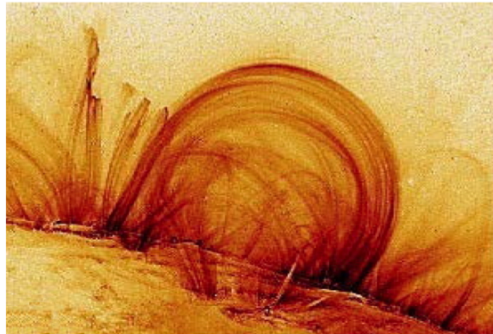
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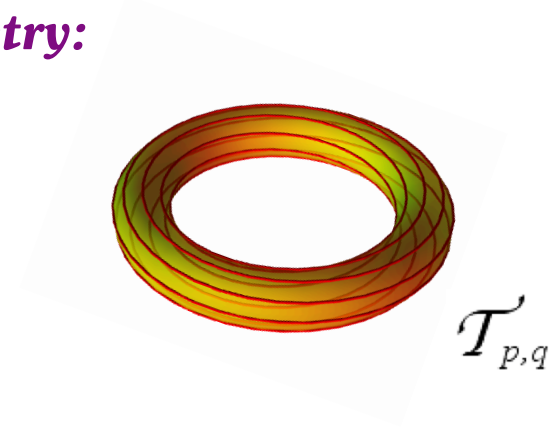
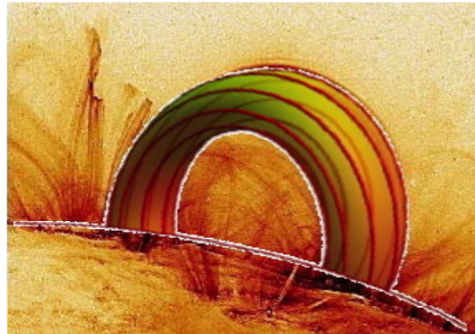
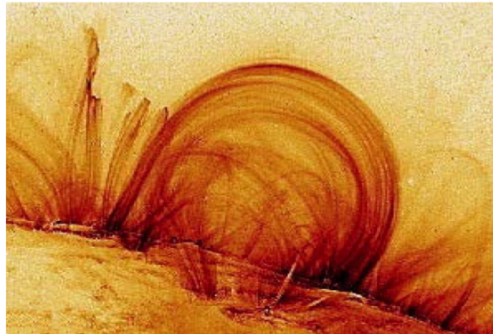
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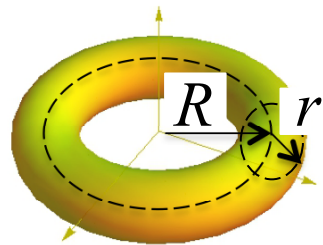
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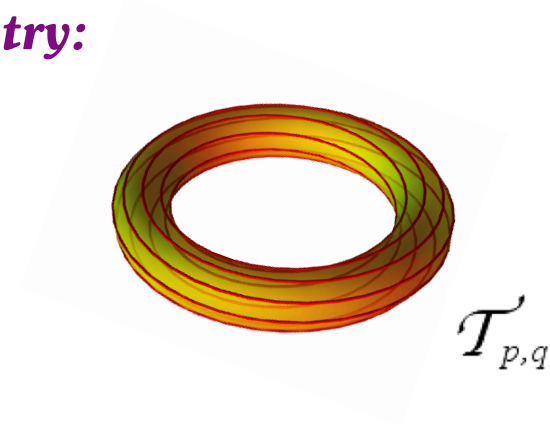
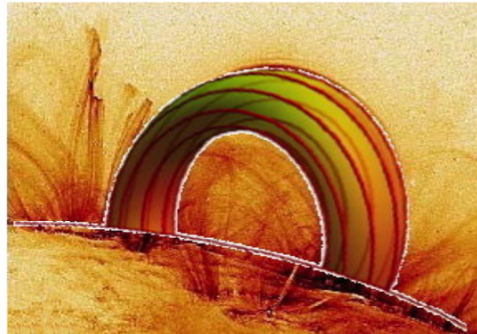
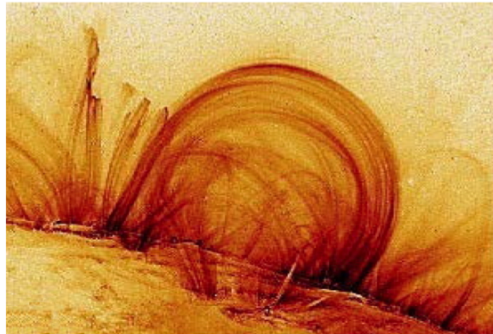
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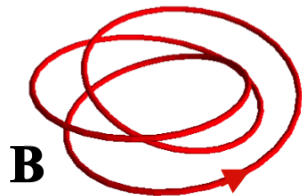
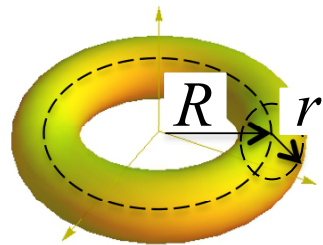
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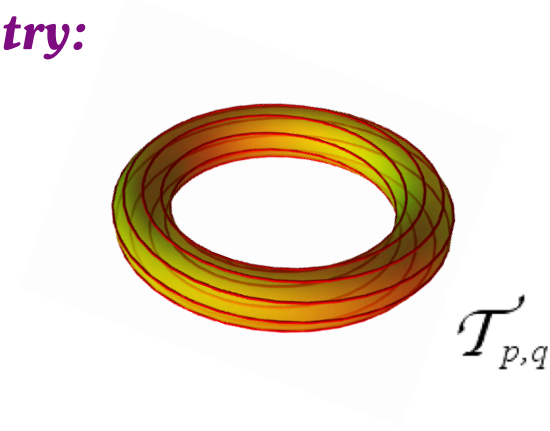
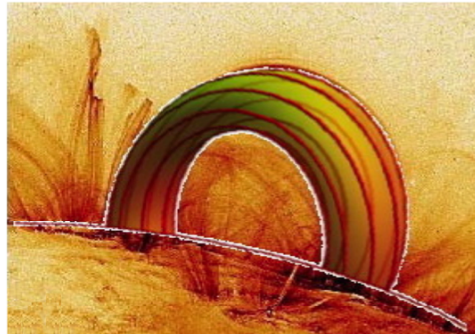
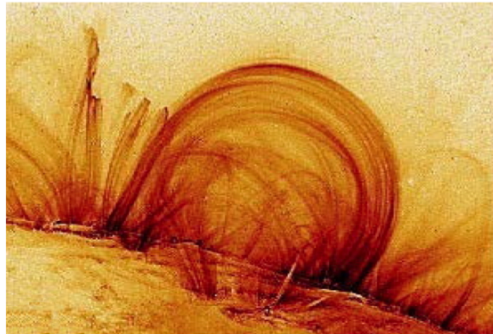
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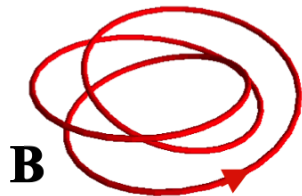
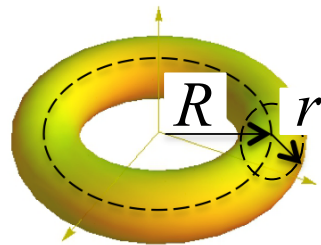
- **Embedding magnetic fields in toroidal geometry:**



- **Magnetic torus knots/unknotted:**

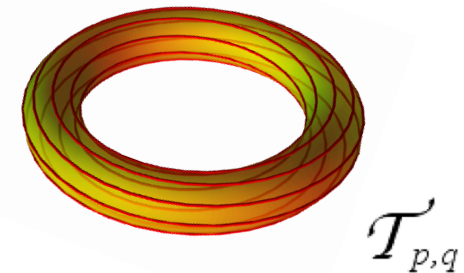
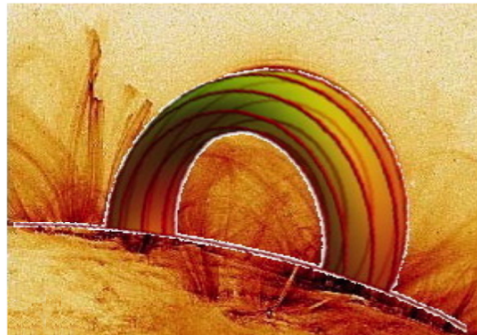
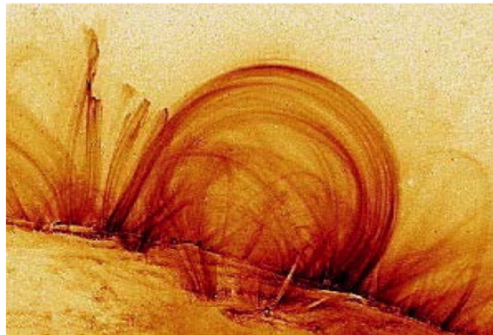
aspect ratio: $\lambda = r/R$

$\mathcal{T}_{p,q} :$ **knots** $\{p, q\} \in \mathbb{N}$ **co-prime;**



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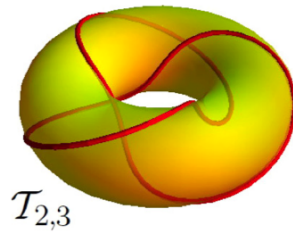
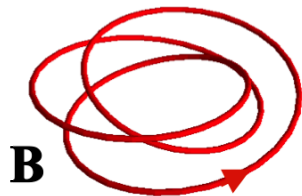
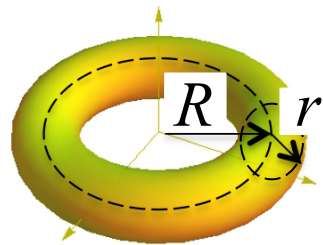
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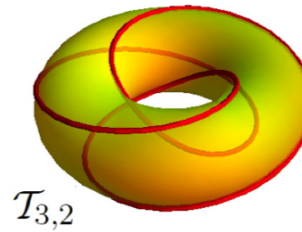
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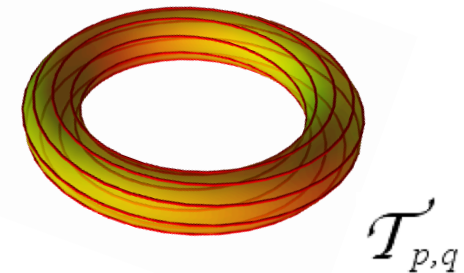
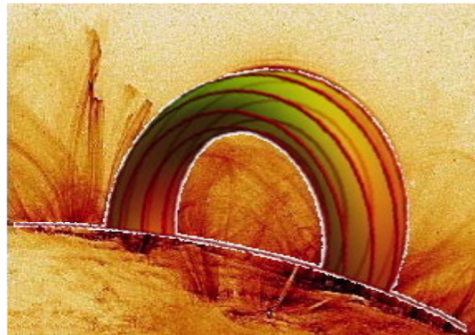
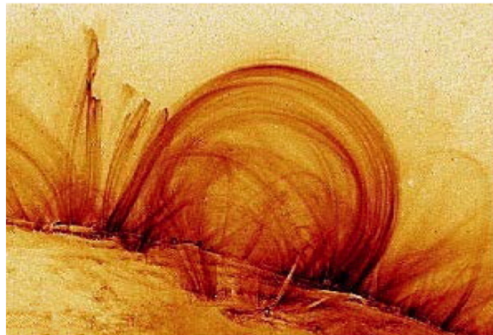


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Force-free magnetic fields in toroidal geometry

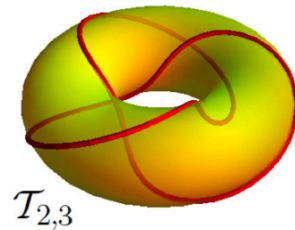
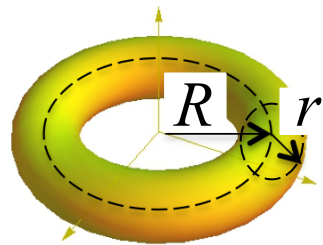
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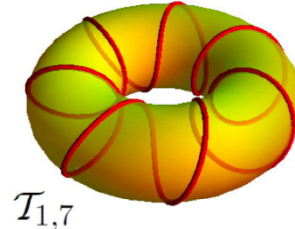
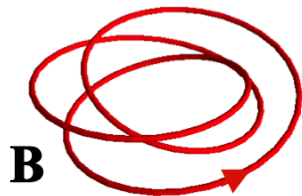
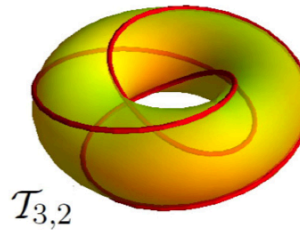
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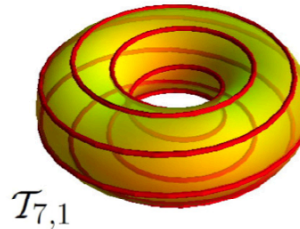
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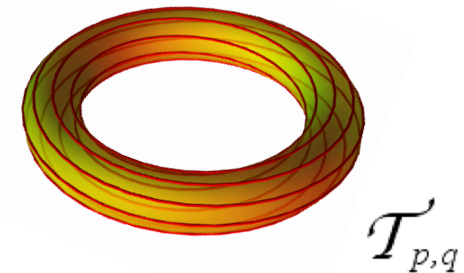
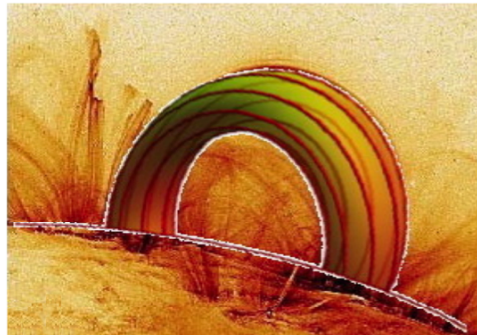
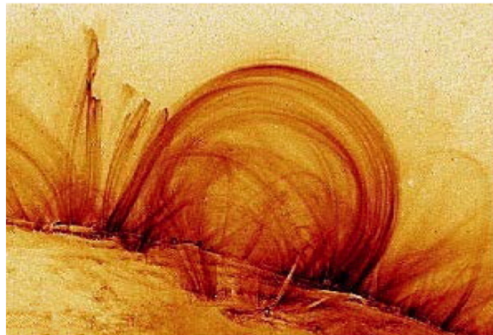


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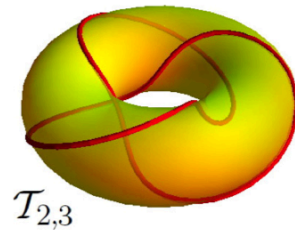
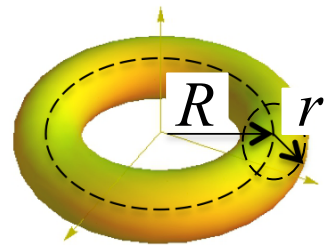
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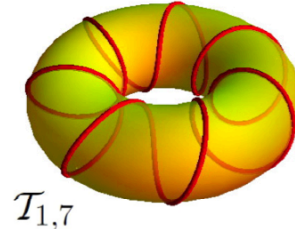
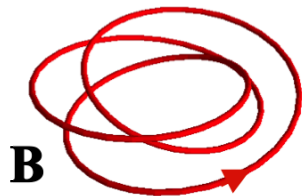
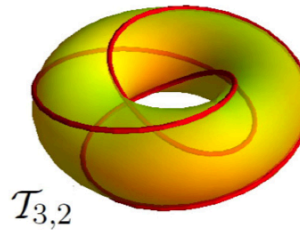
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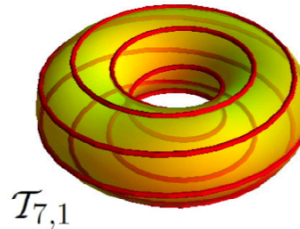
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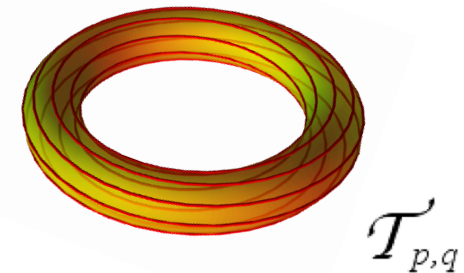
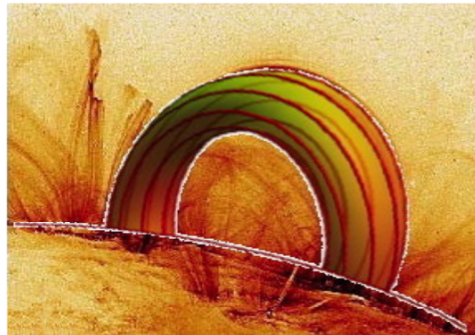
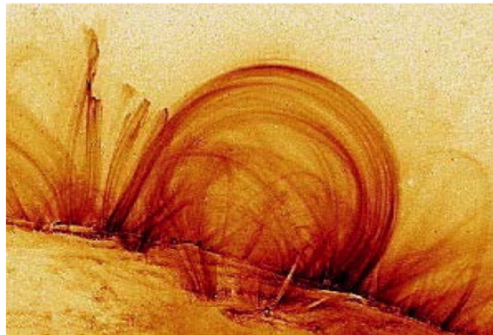
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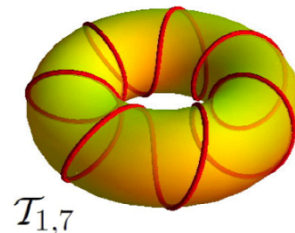
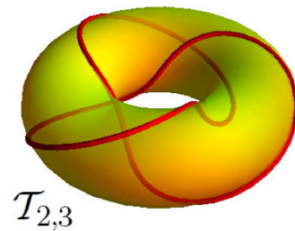
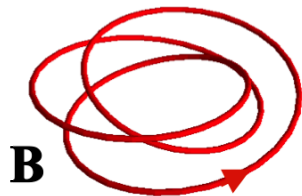
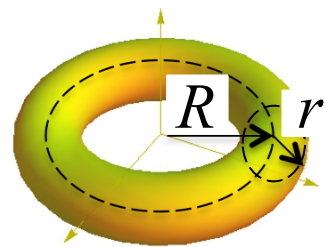
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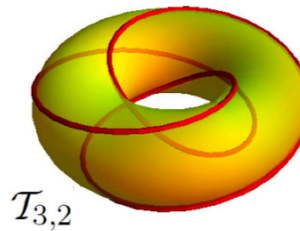
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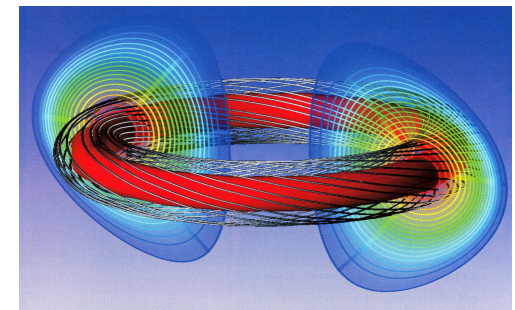
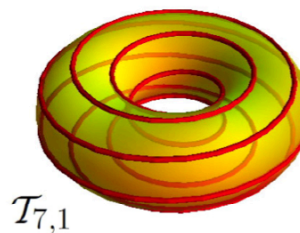
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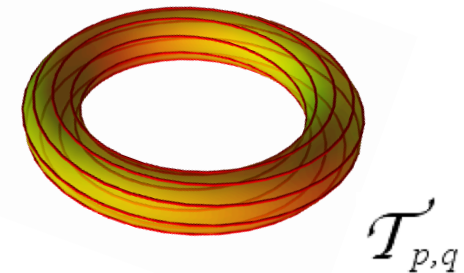
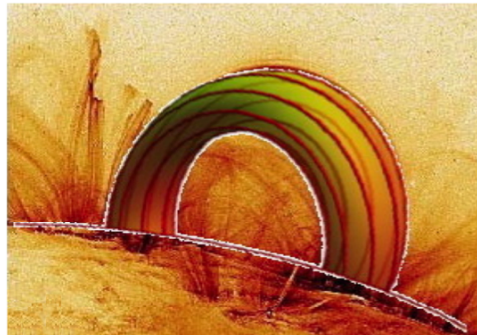
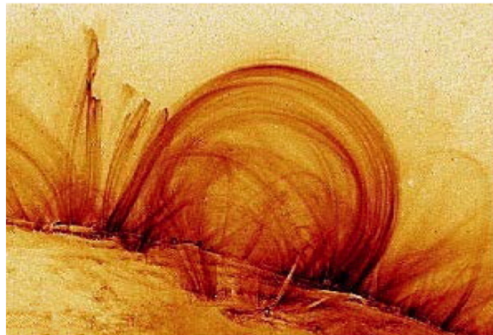
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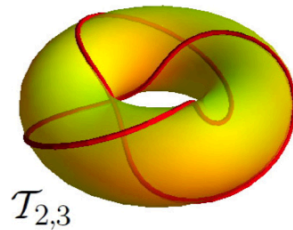
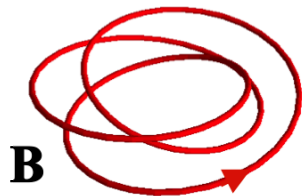
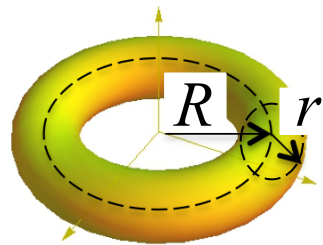
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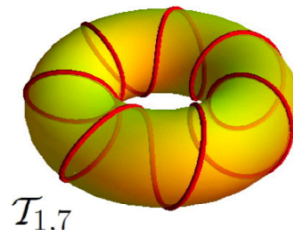
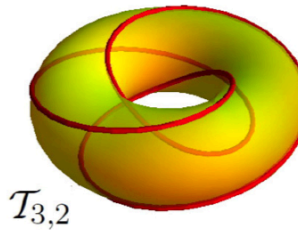
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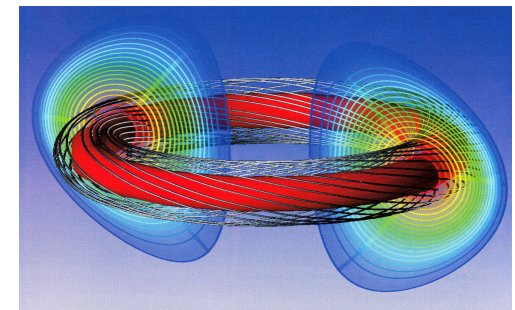
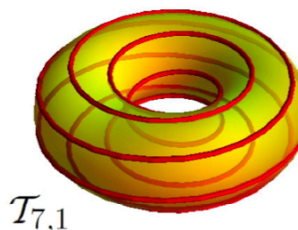
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$\left(\begin{array}{l} \text{tokamak's safety factor: } \chi \propto 1/w = p/q \\ \text{stellarator's rotational transform: } \xi \propto w \end{array} \right)$

Magnetic energy vs. knot complexity

- ***Standard parametrization:***

$$R = 1, \lambda \in (0,1) :$$

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Let $\mathbf{B} = B_0 \hat{\mathbf{t}}$ and $\Phi = 1$; if $\tilde{M} = M/M_o$ is the normalised energy, then:

- **Theorem (Oberti & Ricca 2017).** *For every force-free $\mathcal{T}_{p,q}$ we have:*

$$\frac{p}{2} \left((1 - \lambda)^2 + \lambda^2 w^2 \right)^{1/2} \leq \tilde{M} \leq \frac{p}{2} \left((1 + \lambda)^2 + \lambda^2 w^2 \right)^{1/2}.$$

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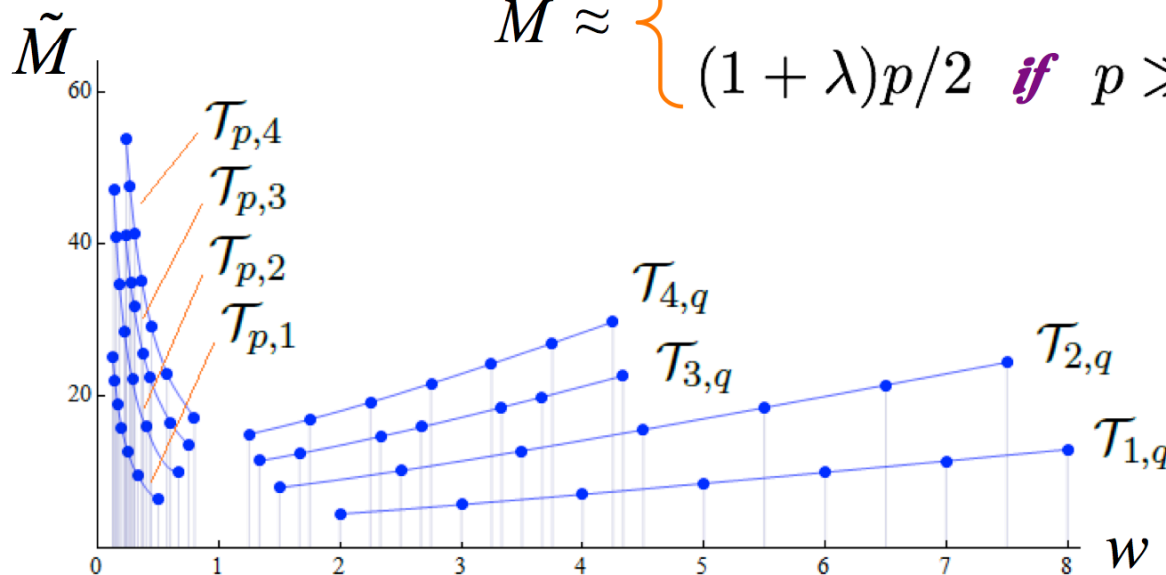
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First 7 torus knots/unknots
for a given topological class

$$\lambda = 0.5$$

Magnetic helicity vs. writhe

- From $\mathbf{B} = \nabla \times \mathbf{A}$ (and $\nabla \cdot \mathbf{A} = 0$) and Biot-Savart we have

$$\mathbf{A}(\mathbf{X}) = \frac{\Phi}{4\pi} \int_0^{2\pi p} \frac{\hat{\mathbf{t}}(\alpha) \times (\mathbf{X} - \mathbf{X}(\alpha))}{|\mathbf{X} - \mathbf{X}(\alpha)|^3} d\mathbf{X}(\alpha) ;$$

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- From the self-linking number decomposition, we have

$$SL = Wr(\mathcal{T}'_{p,q}) + T(\mathcal{T}'_{p,q}), \quad (N = 0).$$

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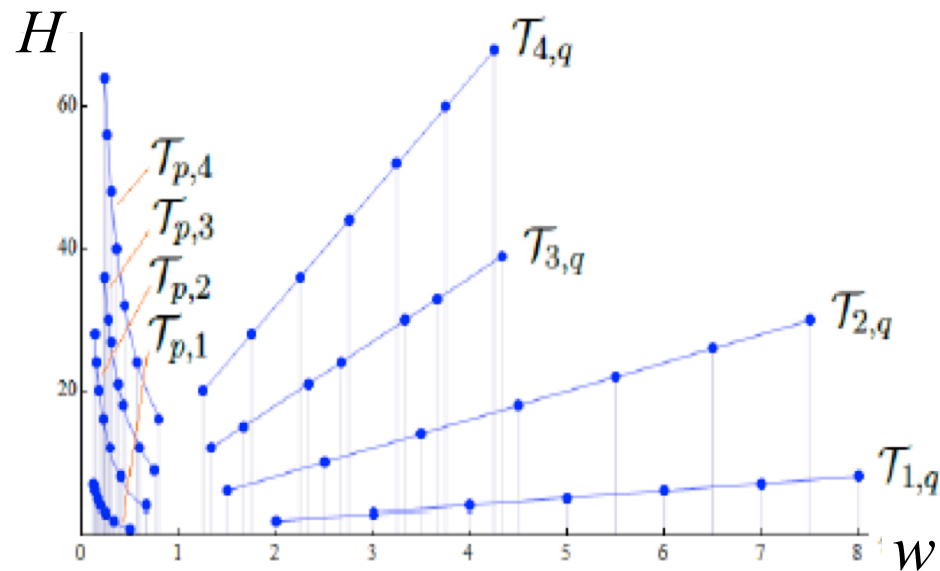
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Direct comparison of H and Wr :



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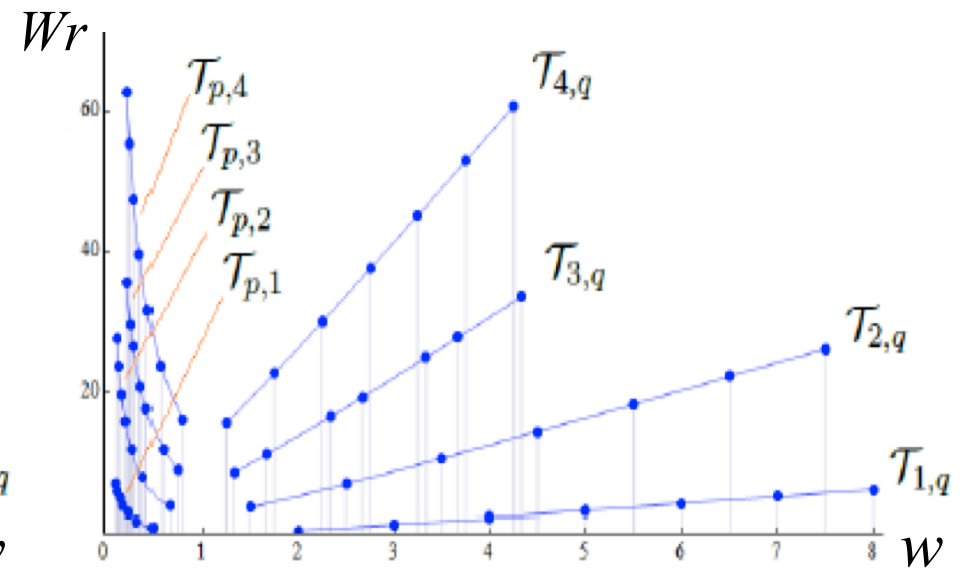
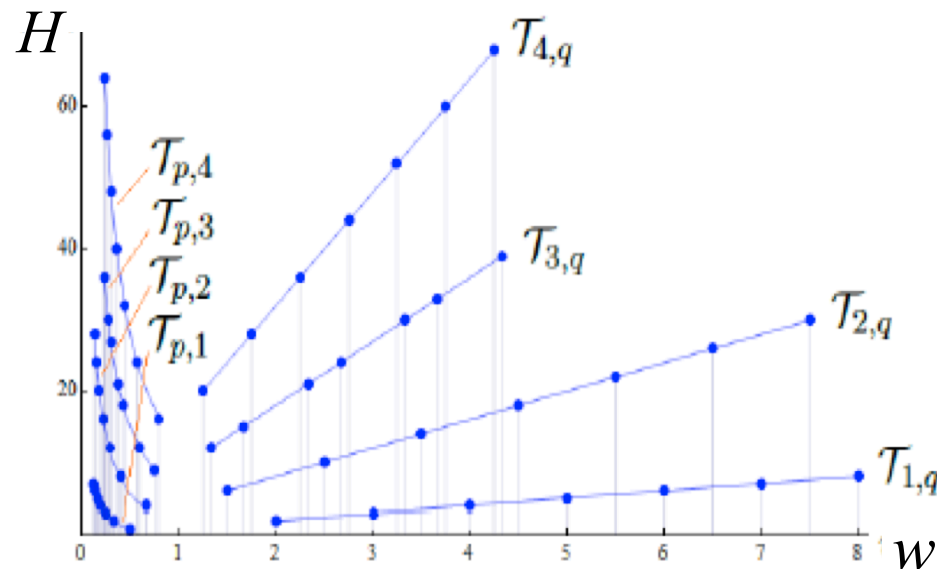
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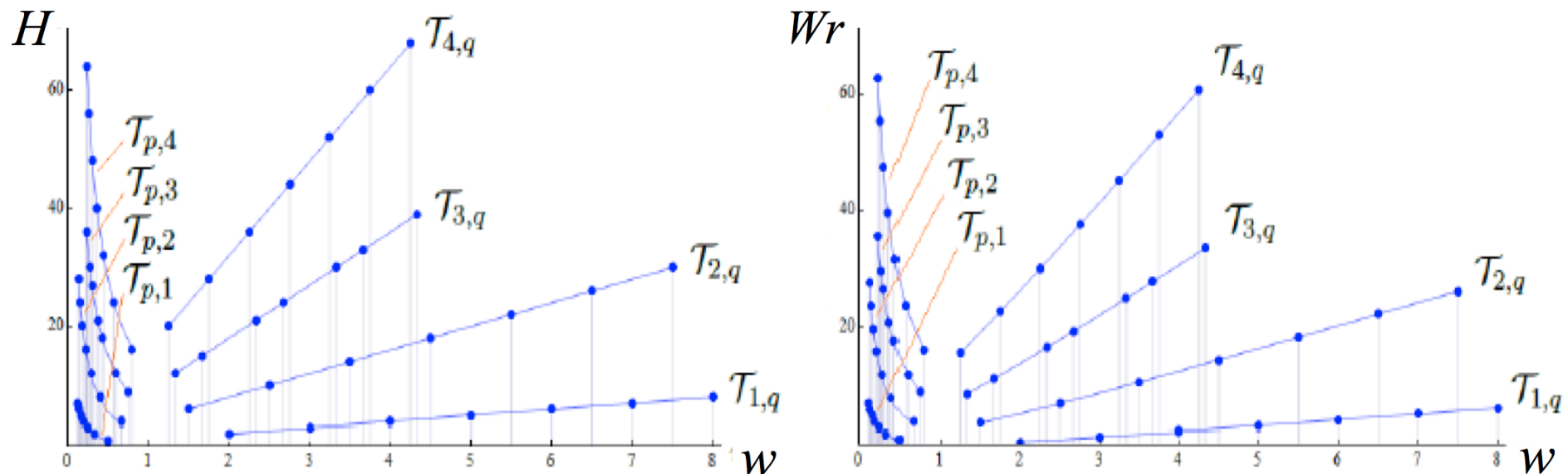
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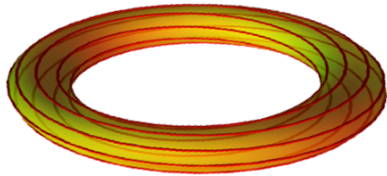
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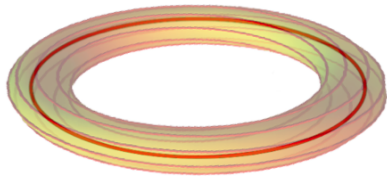


$$H \approx Wr$$

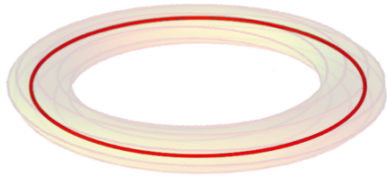
Inflexional states for toroidal fields



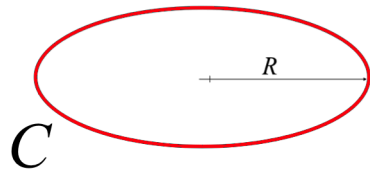
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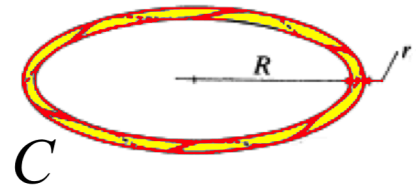
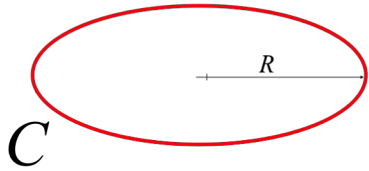
Inflexional states for toroidal fields



Inflexional states for toroidal fields

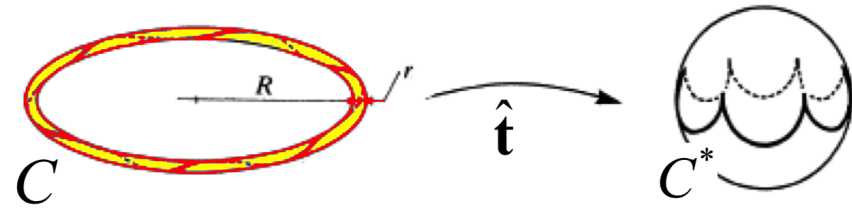
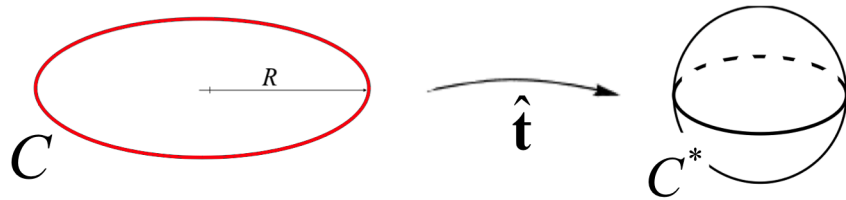


Inflexional states for toroidal fields



Inflexional states for toroidal fields

- *Indicatrix by the tangent map:*



Inflexional states for toroidal fields

- **Indicatrix by the tangent map:**



Theorem (Oberti & Ricca 2016). *Every $\mathcal{T}_{p,q}'$ of winding number w has at least q points of inflexion placed at $\alpha = (2k+1)\pi/w$ ($k = 0, 1, 2, \dots, q-1$) when $\lambda = \lambda_{\text{cr}} = (1+w^2)^{-1}$, or*

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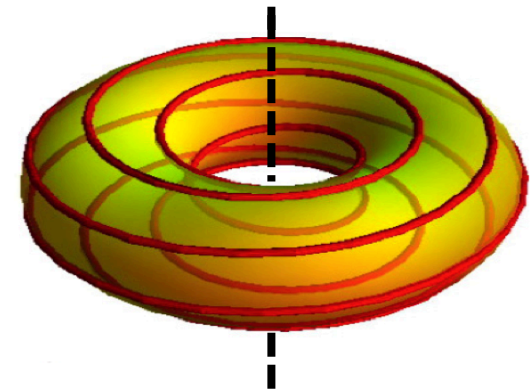
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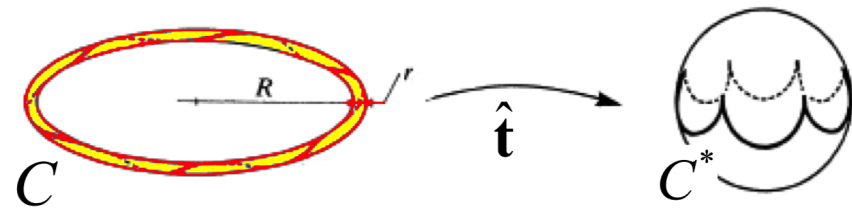
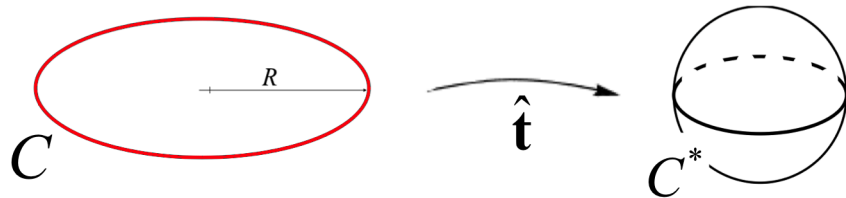
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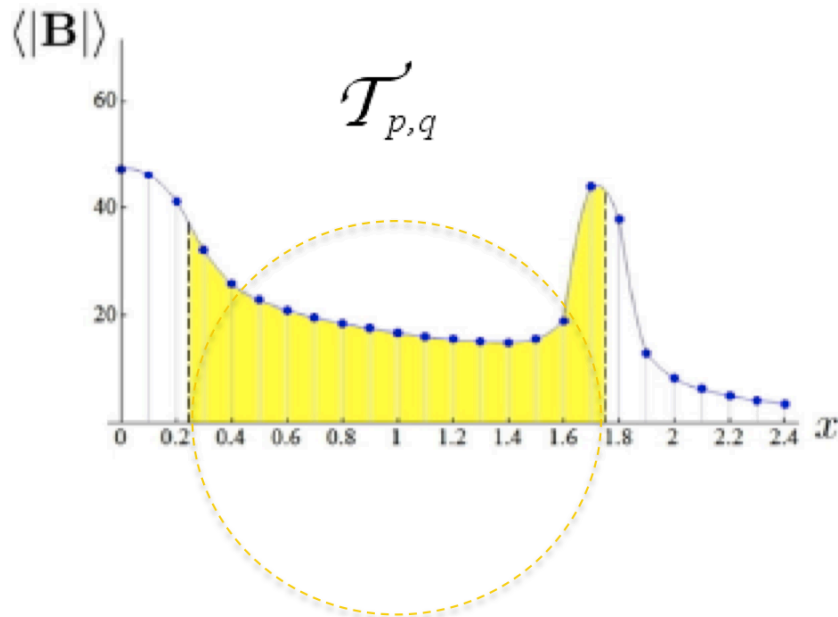
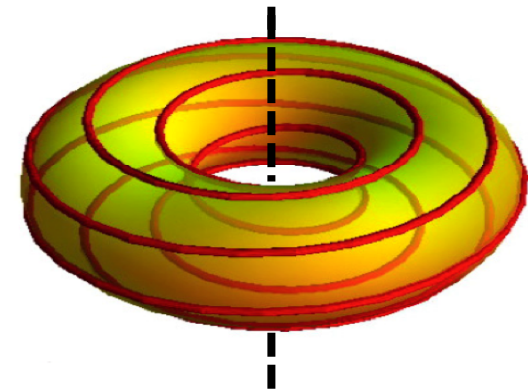
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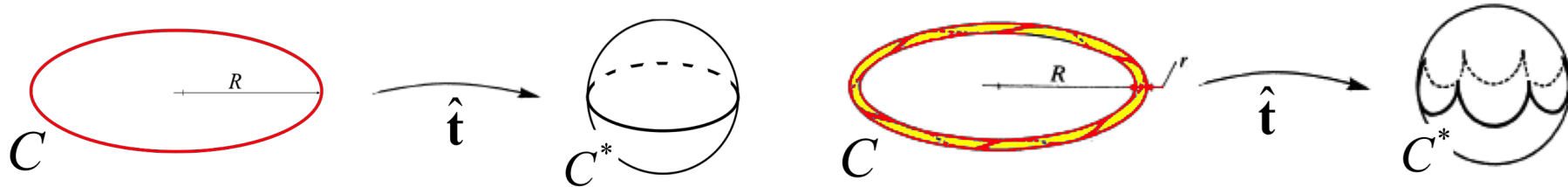
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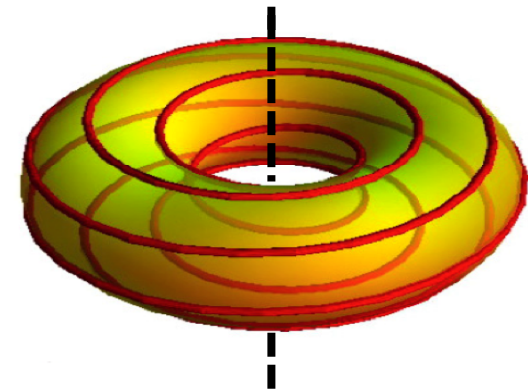
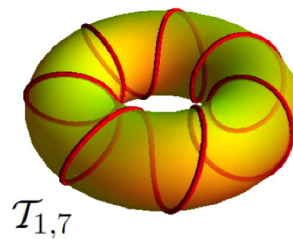
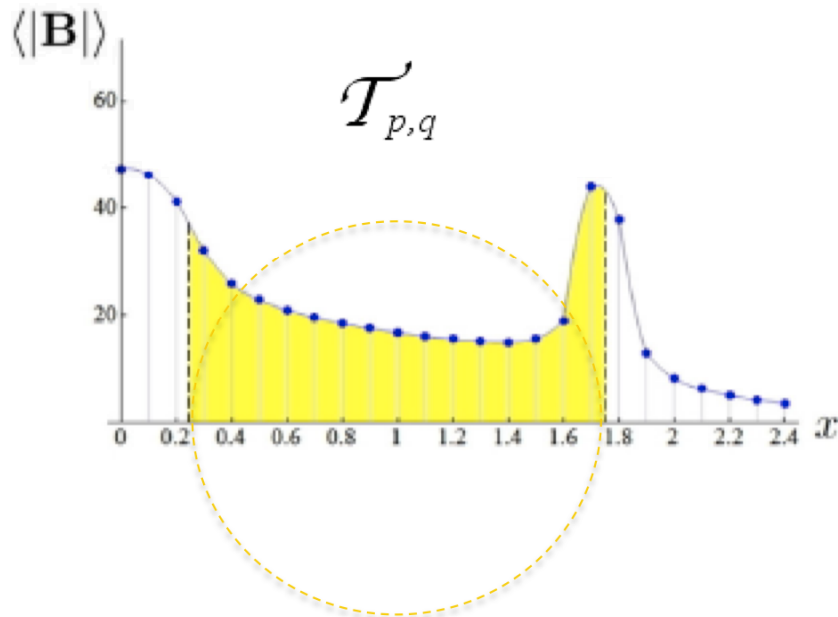
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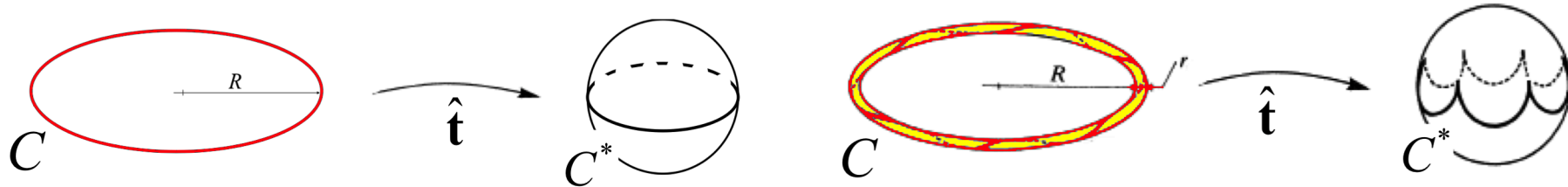


$$p = 1, \quad q = 7 \text{ inflexions}$$

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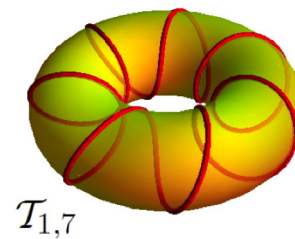
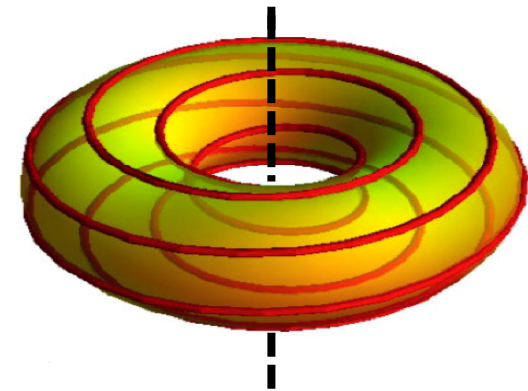
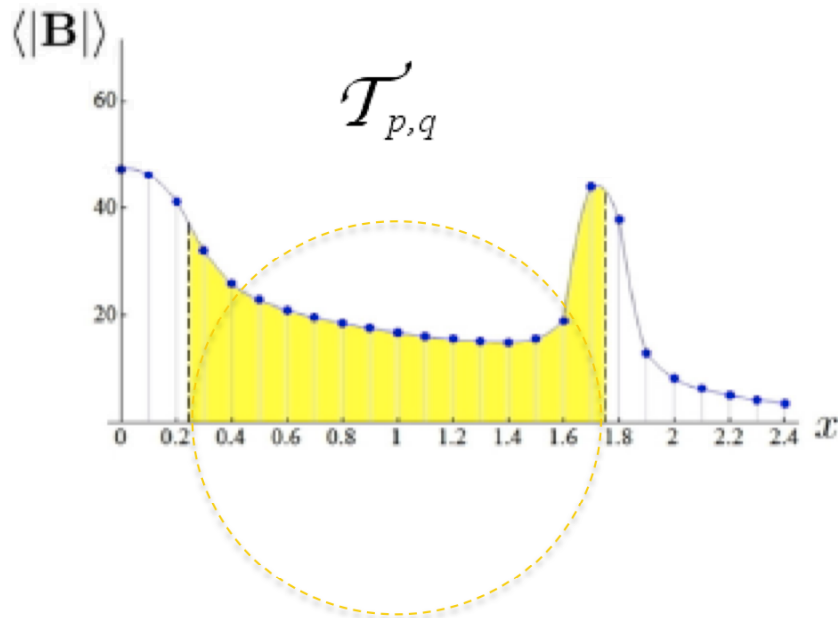
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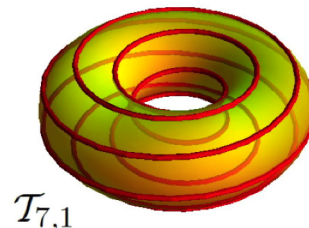


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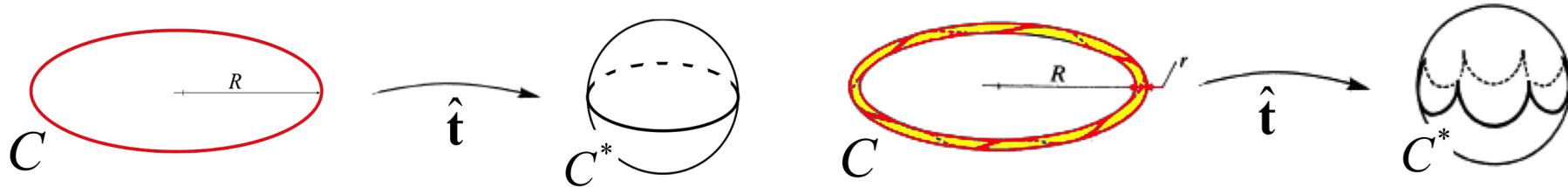
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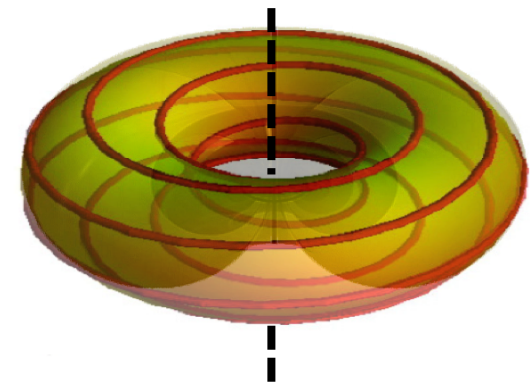
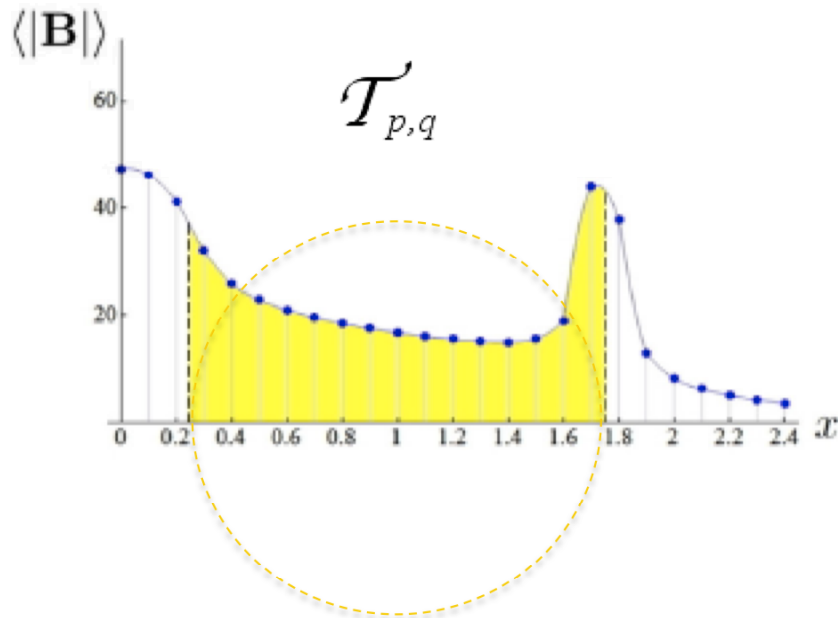
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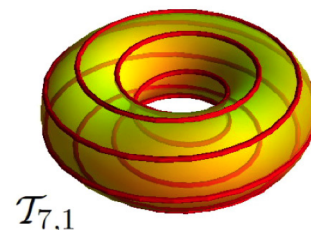
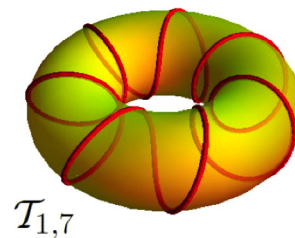


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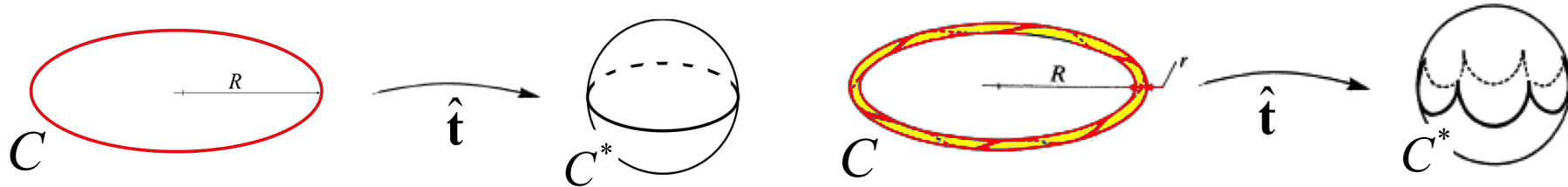
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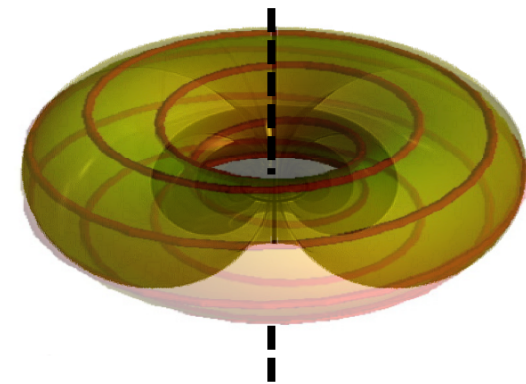
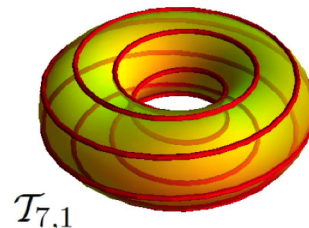
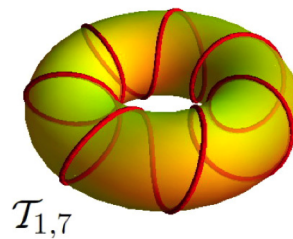
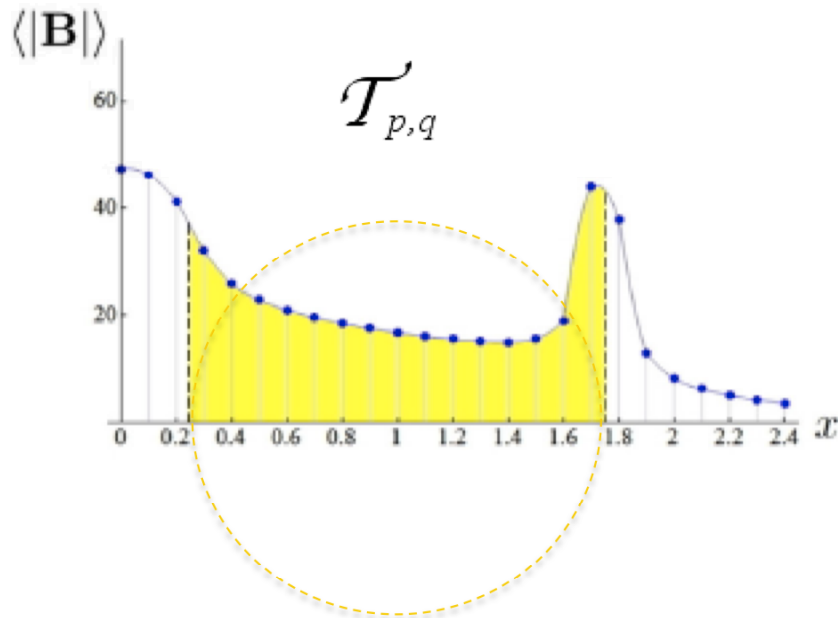
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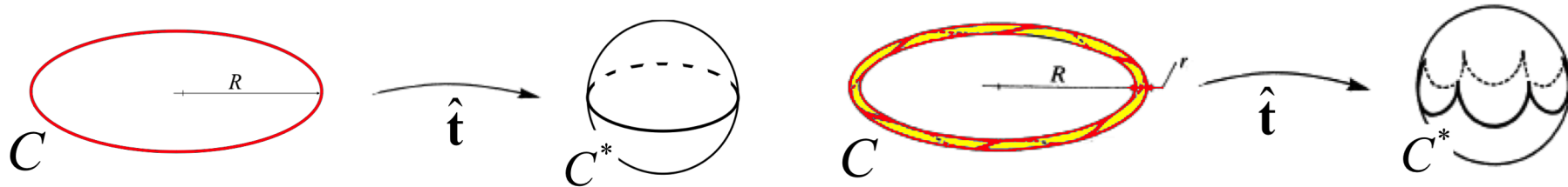


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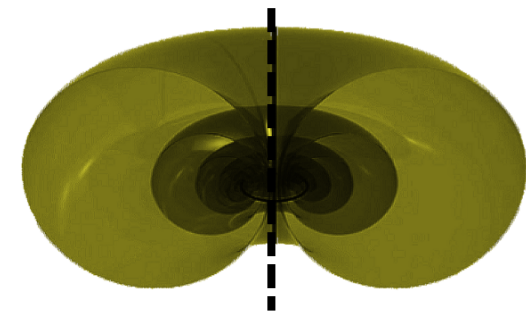
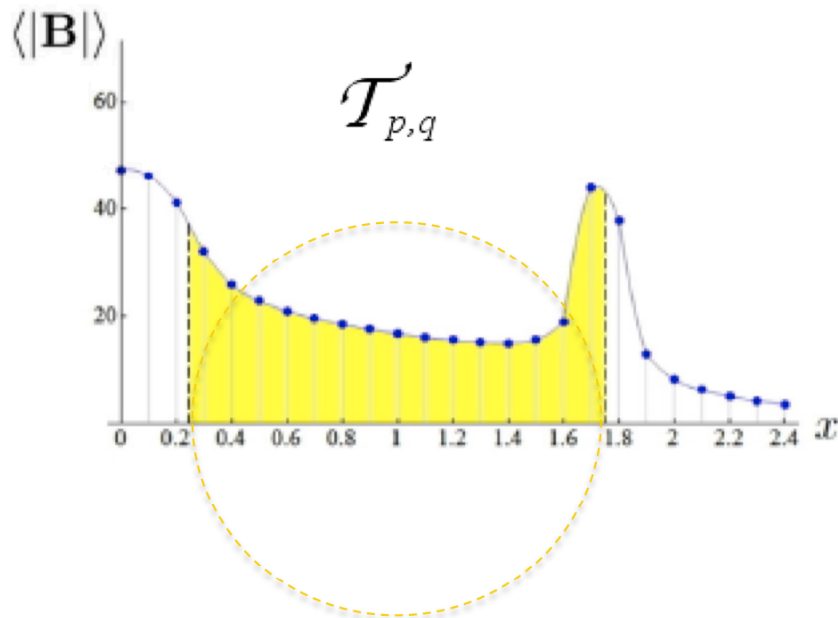
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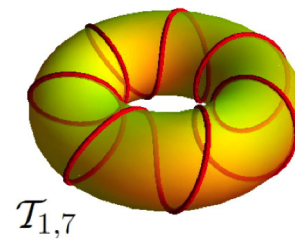


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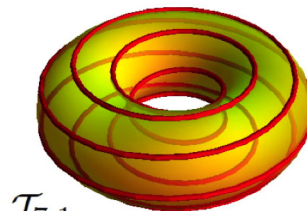


nested tori



$\mathcal{T}_{1,7}$

$p = 1$, $q = 7$ inflexions
 $\alpha = (2k+1)\pi/7$ ($k = 0, 1, \dots, 6$)



$\mathcal{T}_{7,1}$

$p = 7$, $q = 1$ inflexion
 $\alpha = 7\pi$

Multiscale numerical methods in plasma physics

Giacomo Dimarco

Department of Mathematics and Computer Science
Università di Ferrara



UNIVERSITÀ
DEGLI-STUDI
DI-FERRARA

[https://sites.google.com/a/unife.it/giacomo-dimarco-home-page/
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Nicolas Crouseilles (INRIA, Rennes, France), Mohammed Lemou (CNRS, Rennes, France)
Lorenzo Pareschi (University of Ferrara, Italy), Marie-Helene Vignal (University of Toulouse, France)

Outline

- 1 Kinetic models
- 2 The quasi-neutral limit
- 3 Fast methods for the collision operator and time integration

Outline for section 1

- 1 Kinetic models
- 2 The quasi-neutral limit
- 3 Fast methods for the collision operator and time integration

Introduction

- **Kinetic equations** are used to describe a variety of phenomena in different fields : rarefied gas dynamics, plasma physics, biology, socio-economy, etc.
- **They appear** when one considers a **statistical description** of a large particle system evolving in time.
- **At the microscopic level** the particles motion is described by systems of ordinary differential equations.
- **Such systems are extremely costly** to solve numerically and bring **little insight** on the behavior of a large set of particles.
- **Therefore**, one seeks for **reduced models** of the particle dynamics which are still able to describe the physical reality with sufficient accuracy.

Kinetic models

- The Vlasov-Poisson-Landau equation is a kinetic model used to describe long range Coulombian interactions.

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_x \varphi(t, x) \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f), \quad v \in \mathbb{R}^3, x \in \Omega \subset \mathbb{R}^3,$$

where $f(t, x, v)$ depends on time t , position x and velocity v of particles.

- ε is the Knudsen number and $Q(f)$ is the Landau collision operator

$$Q(f)(v) = \nabla_v \cdot \int_{\mathbb{R}^3} \Phi(v - v^*) [\nabla_v f(v) f(v^*) - \nabla_{v^*} f(v^*) f(v)] dv^*.$$

- Φ is a 3×3 nonnegative and symmetric matrix taking the form

$$\Phi(v) = |v|^{\gamma+2} S(v), \quad \gamma \in \mathbb{R} \quad \text{and} \quad S(v) = Id - \frac{v \otimes v}{|v|^2}.$$

- For $\gamma > 0$ we have hard potentials, Maxwellian molecules for $\gamma = 0$ Maxwellian molecules and soft potentials for $\gamma < 0$. The latter case involves the Coulombian case $\gamma = -3$.

Quasi neutral Vlasov-Poisson system

- The electric potential φ is coupled to f through the Poisson equation

$$\Delta\varphi = \frac{e}{\epsilon_0}(\rho - 1), \quad \text{with} \quad \rho = \int f dv.$$

where e is the electric charge and ϵ_0 is the vacuum permittivity. A classical rescaling of the Vlasov-Poisson system leads to

$$\gamma^2 \Delta\varphi = \rho - 1, \quad \text{with} \quad \rho = \int f dv.$$

- Where we denoted by $\gamma = \left(\frac{\epsilon_0 k_B T_0}{e^2 n_0} \right)^{1/2}$ the scaled Debye length, with k_B the Boltzmann constant, with n_0 the plasma density scale and T_0 the plasma temperature scale.
- The numerical solution of a kinetic equation involves several problems of different nature.

Numerical challenges

Aside from the **high dimensionality of the problem**, in general $(x, v, t) \in \mathbb{R}^7$, some of the additional numerical difficulties and requirements specific to kinetic equations are :

- **Conservation properties.** Physical conservation properties are very important since they characterize the steady states.
- **Computational cost.** The operator $Q(f)$ may be described by a high dimensional integral in velocity space at each point x in physical space.
- **Velocity range.** The significant velocity range may vary strongly with space position (steady states are not compactly supported in velocity space).
- **Presence of multiple scales.** In presence of multiple space-time scales and/or large velocities the kinetic equation becomes stiff.

Hydrodynamic equations

- Integrating the kinetic equation against the collision invariants leads to

$$\partial_t \langle cf \rangle + \operatorname{div}_x \langle v \otimes cf \rangle = 0.$$

- Close to fluid regimes, the mean free path is very small. Thus, passing to the limit $\varepsilon \rightarrow 0$ we obtain $Q(f) = 0$ and $f = M[f]$.
- At least formally, we recover the Euler-Poisson system

$$\partial_t U + \operatorname{div}_x \mathcal{F}(U) = S(U)$$

$$\mathcal{F}(U) = \langle v \otimes cM[f] \rangle = \begin{pmatrix} \rho u \\ \rho u \otimes u + pI \\ Eu + pu \end{pmatrix}, \quad S(U) = \begin{pmatrix} 0 \\ \rho \nabla_x \varphi \\ \rho u \cdot \nabla_x \varphi \end{pmatrix}$$

with $U = \langle cM[f] \rangle = (\rho, \rho u, E)^T$.

Outline for section 2

- 1 Kinetic models
- 2 The quasi-neutral limit
- 3 Fast methods for the collision operator and time integration

Quasi neutral Vlasov Poisson system

- We consider the so-called collisional Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f).$$

when $\gamma \rightarrow 0$.

- In this case we get from the Poisson equation

$$\gamma \Delta \varphi = (\rho - 1), \quad \rho \rightarrow 1$$

which means that we lose an equation for the electric potential.

- In order to recover an equation for the potential φ , we assume that the quasineutrality constraint is satisfied initially
- We derive with respect to time the continuity equation

$$\partial_{tt} \rho + \partial_t \nabla_x \cdot (\rho u) = 0.$$

The reformulated quasi neutral Vlasov Poisson system

- Then, taking the divergence of momentum equation

$$\nabla_x \cdot \partial_t(\rho u) + \nabla_x^2 : S = \nabla_x \cdot (-\rho \nabla_x \varphi)$$

where $S = \int f v \otimes v dv$.

- Making the difference between the above two equations

$$\partial_{tt}\rho - \nabla_x^2 : S = \nabla_x \cdot (\rho \nabla_x \varphi).$$

- Finally, using the Poisson equation to replace gives the Reformulated Poisson Equation (RPE)

$$-\gamma^2 \partial_{tt} \Delta \varphi - \nabla_x^2 : S = \nabla_x \cdot (\rho \nabla_x \varphi).$$

which is equivalent to the original one if initially the Poisson equation and its time derivative are satisfied.

The limit systems

- Thus the reformulated system reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla \varphi \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f),$$

$$-\gamma^2 \partial_{tt} \Delta \varphi - \nabla_x^2 : S = \nabla_x \cdot (\rho \nabla_x \varphi)$$

- The quasi-neutral limit of Vlasov-Poisson system reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla \varphi \cdot \nabla_v f = \frac{1}{\varepsilon} Q(f),$$

$$-\nabla_x^2 : S = \nabla_x \cdot (\rho \nabla_x \varphi).$$

- The Reformulated Vlasov-Poisson system in the fluid limit reads

$$\partial_t U + \nabla_x \cdot F(U) = G(U),$$

$$-\gamma^2 \partial_{tt} \Delta \varphi - \nabla_x^2 : S = \nabla_x \cdot (\rho \nabla_x \varphi).$$

- The Euler-Poisson quasi-neutral system reads

$$\partial_t U + \nabla_x \cdot F(U) = G(U),$$

$$-\nabla_x^2 : S = \nabla_x \cdot (\rho \nabla_x \varphi).$$

A first order splitting

The initial condition satisfies the quasi-neutral limit: $\rho^n = 1$ and $\nabla_x \cdot (\rho u)^n = 0$

$$\begin{aligned} f^* &= f^n - \Delta t \nabla_x \varphi \cdot \nabla_v f^n, \\ f^{n+1} &= f^* - \Delta t v \cdot \nabla_x f^*. \end{aligned}$$

This gives for the density and the momentum

$$\begin{aligned} \rho^* &= \rho^n = 1, \\ (\rho u)^* &= (\rho u)^n + \Delta t \rho^n \nabla_x \varphi, \end{aligned}$$

so that $\nabla_x \cdot (\rho u)^* = \Delta t \nabla_x \cdot (\rho^n \nabla_x \varphi)$. The second step gives

$$\begin{aligned} \rho^{n+1} &= \rho^* - \Delta t \nabla_x \cdot (\rho u)^* = \rho^n - \Delta t^2 \nabla_x \cdot (\rho^n \nabla_x \varphi), \\ (\rho u)^{n+1} &= (\rho u)^* - \Delta t \nabla_x \cdot S^*. \end{aligned}$$

First order splitting and high order splitting

And, then

$$\nabla_x \cdot (\rho u)^{n+1} = \nabla_x \cdot (\rho u)^* - \Delta t \nabla_x^2 : S^* = \Delta t (\nabla_x \cdot (\rho^n \nabla_x \varphi) - \nabla_x^2 : S^*).$$

- This means that in principle, we can choose an electric potential which ensures that $\nabla_x \cdot (\rho u)^{n+1} = 0$,
- This is the case if φ is the solution of $\nabla_x \cdot (\rho^n \nabla_x \varphi) = \nabla_x^2 : S^*$.
- However, in the general case, there is no choice of the electric potential which ensures the propagation of the quasi-neutral state $\rho^{n+1} = \rho^n = 1$.
- If we repeat the same analysis for a second order splitting as for instance the Strang splitting we realize that quasi neutrality is lost for any choice of the initial step.
- There is no easy solution for the construction of high order schemes preserving quasi-neutral states when splitting methods are used.

A new stable scheme for quasi neutrality

Let us first consider the quasi-neutral system and assume quasi-neutral initial conditions *i.e.* $\rho^n = 1$ and $\nabla_x \cdot (\rho u)^n = 0$, then we have

$$f^{n+1} = f^n - \Delta t v \cdot \nabla_x f^n - \Delta t \nabla_x \varphi^{n+1} \cdot \nabla_v f^n + \frac{\Delta t}{\varepsilon} Q(f^{n+1}),$$

$$\nabla_x \cdot (\rho^n \nabla_x \varphi^{n+1}) = \nabla_x^2 : S^n.$$

Taking the velocity moments leads to

$$\begin{aligned} \rho^{n+1} &= \rho^n - \Delta t \nabla_x \cdot (\rho u)^n, \\ (\rho u)^{n+1} &= (\rho u)^n - \Delta t \nabla_x \cdot S^n + \Delta t \rho^n \nabla_x \varphi^{n+1}, \end{aligned}$$

which gives

$$\rho^{n+1} = \rho^n, \quad \text{and} \quad \nabla_x \cdot (\rho u)^{n+1} = -\Delta t (\nabla_x^2 : S^n - \nabla_x \cdot (\rho^n \nabla_x \varphi^{n+1})) = 0.$$

Thus quasi-neutrality constraint is propagated in time.

Reformulation of the new scheme

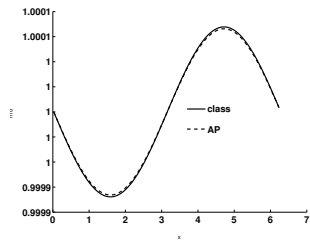
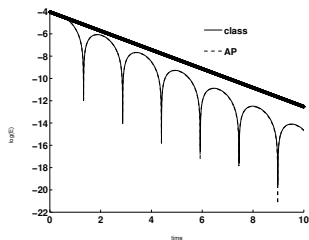
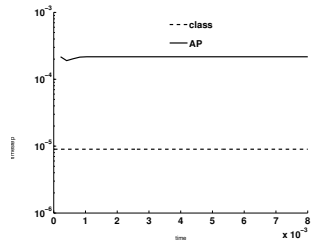
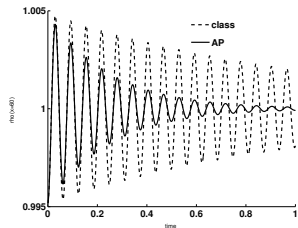
- The same scheme in the general case reads

$$f^{n+1} = f^n - \Delta t v \cdot \nabla_x f^n - \Delta t \nabla_x \varphi^{n+1} \cdot \nabla_v f^n + \frac{\Delta t}{\varepsilon} Q(f^{n+1}),$$

$$\nabla_x \cdot [(\lambda^2 + \Delta t^2 \rho^n) \nabla_x \varphi^{n+1}] = \rho^{n+1} - 1 - \Delta t \nabla_x \cdot (\rho u)^n + \Delta t^2 \nabla_x^2 : S^n.$$

- If we let $\lambda \rightarrow 0$ and if the initial data are consistent with the quasi neutral limit then the scheme is consistent with the quasi neutral limit.
- If $\lambda = 0$ at $t = 0$, but the initial data are not consistent with the quasi neutral limit we get after two time step that the solution is projected over the quasineutrality.
- If at a given instant of time t^n , the Debye length becomes zero, at the time step t^{n+2} the quasi neutrality is obtained and then propagated for all times.
- A linear stability result proves that the scheme proposed is stable for all values of λ to small perturbations of the quasi neutral equilibrium state.

Landau damping



Outline for section 3

- 1 Kinetic models
- 2 The quasi-neutral limit
- 3 Fast methods for the collision operator and time integration

A spectral method for the collision operator

- We approximate the distribution by a partial sum of a Fourier series,

$$f_N(t, v) = \sum_{k \in \{-N, \dots, N\}} \hat{f}_k(t) e^{i k \cdot v},$$

where the k -th mode is given by

$$\hat{f}_k(t) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} f(t, v) e^{-i k \cdot v} dv.$$

- Substituting the approximation $f_N(t, v)$ in the Landau operator gives

$$Q^R(f_N, f_N) = \sum_{k \in \{-2N, \dots, 2N\}} \hat{Q}_k^R e^{i k \cdot v},$$

with

$$\hat{Q}_k^R = \sum_{\substack{l+m=k \\ l, m=-N}}^N \hat{f}_l \hat{f}_m \hat{\beta}(l, m),$$

where $\hat{\beta}(l, m) = \hat{B}(l, m) - \hat{B}(m, m)$ and

A spectral method for the collision operator II

$$\hat{B}(l, m) = \int_{\mathcal{B}_0(\pi)} |g|^{\gamma+2} \left[l^2 - \left(l \cdot \frac{g}{|g|} \right)^2 \right] e^{ig \cdot m} dg.$$

- Let now $\mathcal{P}_N : L^2([-\pi, \pi]^d) \rightarrow \mathbb{P}^N$ be the orthogonal projection upon the space of trigonometric polynomials of degree N in v .
- Then the spectral method for the collision part can be written as

$$\frac{\partial f_N}{\partial t} = \frac{1}{\varepsilon} Q_N^L(f_N, f_N)$$

with initial data $f_N(v, t=0) = f_{0,N}(v)$ and

$$Q_N^L(f_N, f_N) := \mathcal{P}_N Q^R(f_N, f_N),$$

- It is easy to verify that the spectral method preserves mass whereas variations of momentum and energy are controlled by the spectral accuracy.

A fast summation method

- A direct computation of $Q^R(f_N, f_N)$ has the same $O(n^2)$, $n = N^3$, cost of a conventional finite difference discretization applied to the Landau equation.
- On the other hand we have

$$\hat{Q}_k^R = \sum_{m=-N}^N \hat{f}_{k-m} \hat{f}_m \hat{B}(k-m, m) - \sum_{m=-N}^N \hat{f}_{k-m} \hat{f}_m \hat{B}(m, m),$$

- Clearly the **second sum is a convolution** sum and thus transform methods allow this term to be evaluated in $O(n \log_2 n)$ operations.
- In the case of the Landau equation, $\hat{B}(l, m)$ splits as

$$\hat{B}(l, m) := l^2 \tilde{F}(m) - \sum_{p,q=1}^d l_p l_q I_{pq}(m) = l^2 \tilde{F}(m) - l \mathcal{I}(m) l^T,$$

where l^T denotes the transpose of the vector l , $\mathcal{I} = (I_{pq})$ is a $d \times d$ symmetric matrix and

A fast summation method II

$$\tilde{F}(m) = \int_{\mathcal{B}_0(\pi)} |g|^{2+\gamma} e^{ig \cdot m} dg,$$

$$I_{pq}(m) = \int_{\mathcal{B}_0(\pi)} |g|^\gamma g_p g_q e^{ig \cdot m} dg, \quad p, q = 1, \dots, d.$$

- The resulting scheme requires the evaluation of $2d + 2$ convolution sums: the number of distinct elements of $\mathcal{I}$ plus two single convolution sums for $\tilde{F}(m)$ and $\hat{B}_L(m, m)$.
- The overall cost of the scheme is only $O(n \log_2 n)$.
- For the implementation of the algorithm we need to evaluate the quantities $\tilde{F}(m)$ and $I_{pq}(m)$.
- One can show that this computation reduces simply to the computation of two one dimensional integrals which can be computed very accurately and then stored.

Main goal of the time discretization

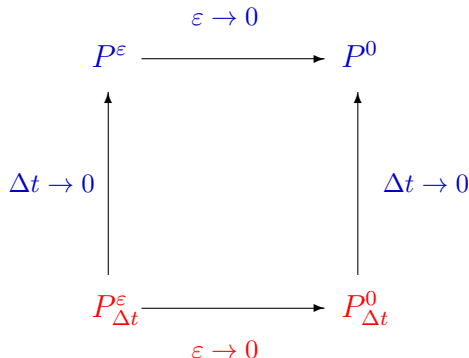
The goal is to construct simple and efficient time discretizations for the solution of kinetic equations in regions with a large variation in the mean free path.

Requirements

- For **large Knudsen numbers**, the methods behave as standard explicit methods.
- For **intermediate Knudsen numbers**, the methods are capable to speed up the computation, allowing larger time steps, without degradation of accuracy.
- In the limit of **very small Knudsen numbers**, the collision step replaces the distribution function by the local Maxwellian. This property is usually referred to as **asymptotic preserving (AP)** since it implies consistency with the underlying system of **Euler equations** of gas dynamics.
- The main physical properties are preserved by the schemes. Namely mass, momentum, energy, nonnegativity of the solution and entropy inequality.

In the sequel we will present some recent advances in this direction based on the use of **exponential methods** and **IMEX methods**.

The AP diagram



In the diagram P^ε is the original singular perturbation problem and $P_{\Delta t}^\varepsilon$ its numerical approximation characterized by a discretization parameter Δt .

The *asymptotic-preserving (AP) property* corresponds to the request that $P_{\Delta t}^\varepsilon$ is a consistent discretization of P^0 as $\varepsilon \rightarrow 0$ independently of Δt .

IMEX Formulation

The general formulation of the IMEX schemes for kinetic equations is

$$F^{(i)} = f^n - \Delta t \sum_{j=1}^{i-1} \tilde{a}_{ij} v \cdot \nabla_x F^{(j)} + \Delta t \sum_{j=1}^{\nu} a_{ij} \frac{1}{\varepsilon} Q(F^{(j)})$$

$$f^{n+1} = f^n - \Delta t \sum_{i=1}^{\nu} \tilde{w}_i v \cdot \nabla_x F^{(i)} + \Delta t \sum_{i=1}^{\nu} w_i \frac{1}{\varepsilon} Q(F^{(i)}).$$

$F^{(i)}$ are called stages and f^{n+1} the numerical solution. Using the vector notations

$$\begin{aligned} F &= f^n e + \Delta t \tilde{A} L(F) + \frac{\Delta t}{\varepsilon} A Q(F) \\ f^{n+1} &= f^n + \Delta t \tilde{w}^T L(F) + \frac{\Delta t}{\varepsilon} w^T Q(F), \end{aligned}$$

where $e = (1, 1, \dots, 1)^T \in \mathbb{R}^{\nu}$ and $L(F) = -v \cdot \nabla_x F - \nabla_x \varphi \cdot \nabla_v F$.

Homogeneous test problem I

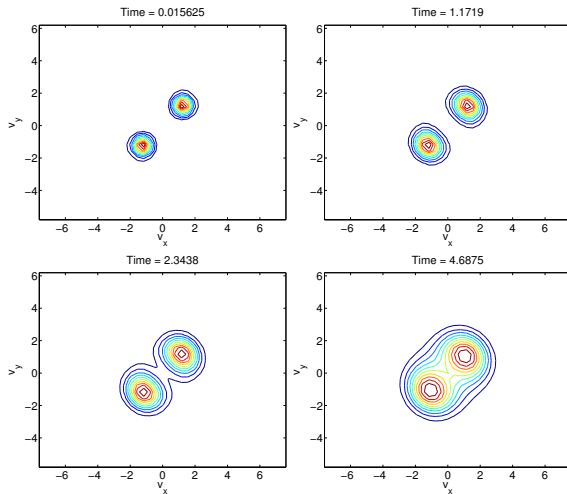


Figure: Time evolution of the level set of distribution function f , starting from a double peak distribution. A first order exponential Runge-Kutta method is used for the time integration.

Homogeneous test problem II

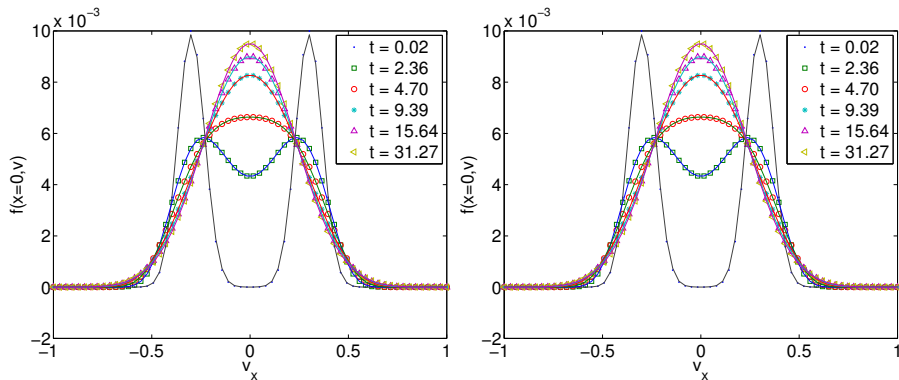


Figure: Time evolution of distribution function f at various time slot for Rosenbluth's test problem. Solid lines: reference solution. Markers: solution computed by exponential RK3 method (left) and third order IMEX-RK method (right).

Convergence rates

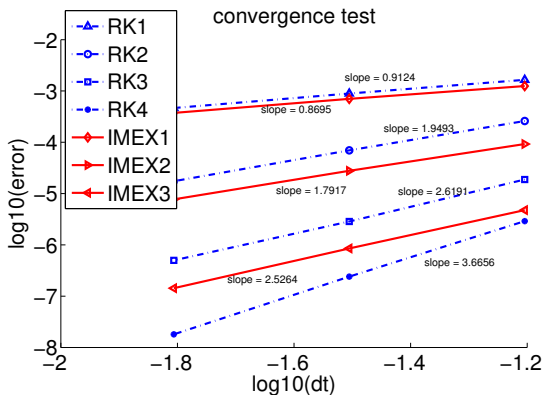


Figure: The convergence rates of the exponential Runge-Kutta methods (dotted line with markers) and of the IMEX-RK method (solid line with markers) for the homogeneous Landau-Fokker-Planck equation.

Two stream instability

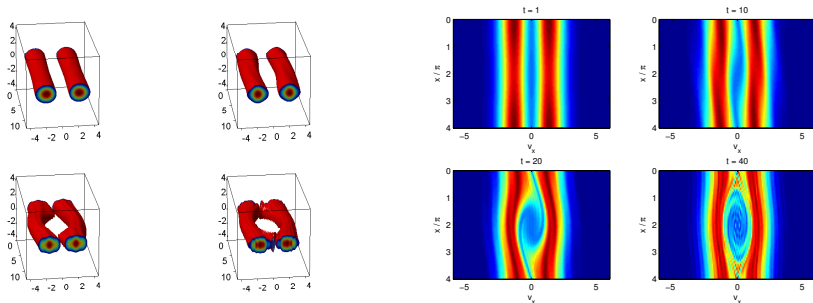


Figure: Two stream instability at different times. Left: 3D illustration; Right: projection in the $x - v_x$ plane.

Thanks!

inverse problems for solar plasma spectroscopy, imaging and imaging spectroscopy

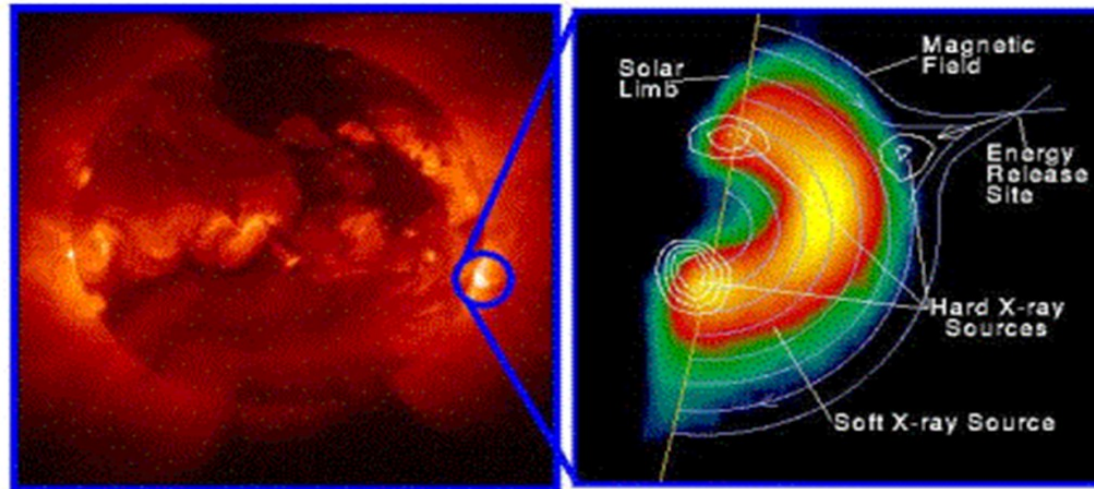
fisica del plasma e matematica applicata
il ruolo della matematica nello sviluppo della fisica del plasma
CNR – roma 6 giugno 2018

michele piana
dipartimento di matematica universita
di genova
CNR – SPIN genova



the solar flare paradox

solar flares



Yohkoh X-ray Image of a Solar Flare, Combined Image in Soft X-rays (left) and Soft X-rays with Hard X-ray Contours (right). Jan 13, 1992.

extend over 10,000 kilometers

release more than 10^{32} ergs in 10-100 seconds

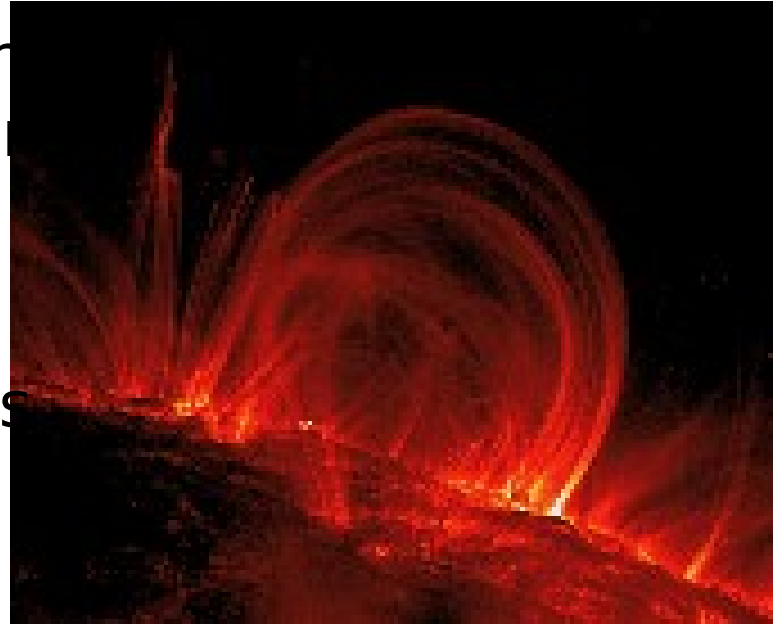
accelerate billion tons of material to more than a million km per second

produce electromagnetic radiation at all wavelengths

energy release and electron acceleration

es result from the release of energy
red in magnetic field configuration
ociated with active regions

energy release accelerates electrons
a rate of 10^{37} s^{-1}



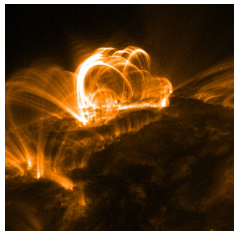
celeration is revealed mainly by hard X-rays ($> 10 \text{ keV}$)
mstrahlung between accelerated electrons and plasma ions

impossible numbers

- rate of e- acceleration in large flare $\sim 10^{37} \text{ s}^{-1}$
 - associated current $\sim 10^{37} \text{ e}^- \text{ s}^{-1} \sim 10^{18} \text{ A}$
 - width of channel $\sim 10^7 \text{ m}$
 - ampère law $\sim B = \mu_0 I / 2\pi r \sim 10^4 \text{ T}$
 - faraday law $\sim V = L \, dI/dt \sim (\mu_0 \ell) I / \Delta \sim 10^{19} \text{ V}$
- these numbers are impossibly large



high resistance + low inductance =
short time for current to flow



low resistance + high inductance =
(impossibly) long time for current to excite the

space instruments at genova

- RHESSI team
- two co-Is for STIX in solar orbiter
- two co-Is for FOXSI
- FLARECAST

reuve ramaty high energy solar spectroscopic imager (RHESSI)

launched on feb 5 2002

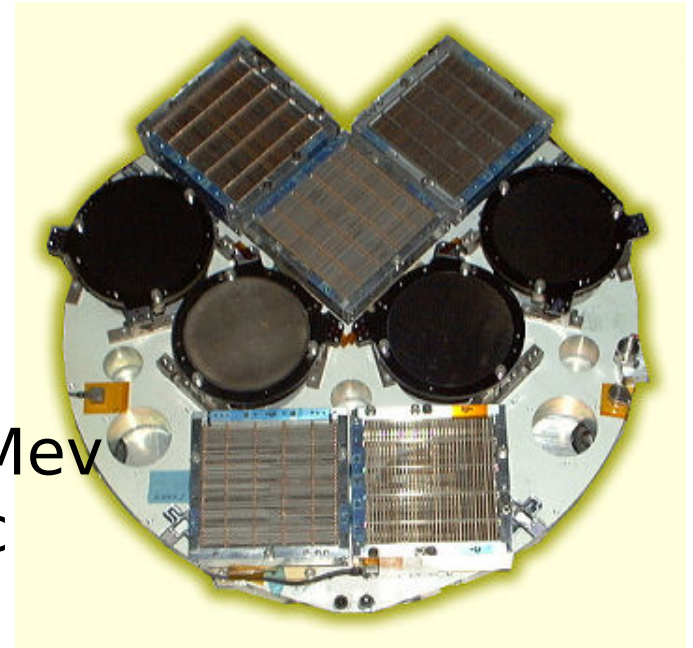
9 Ge detectors

9 rotating modulation collimators

spectral resolution: around 1 keV

spectral range: a few keV to a few MeV

imaging resolution: up to 2.3 arcsec

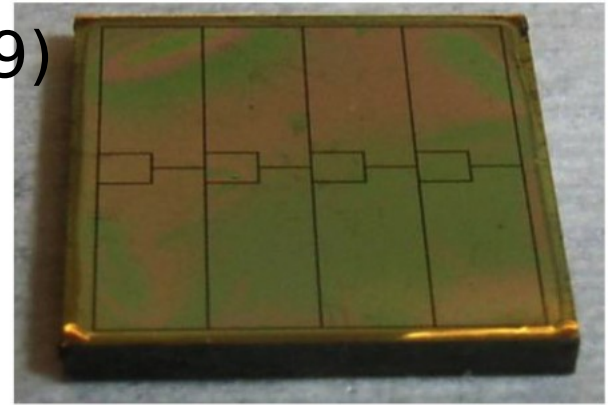


se problems for imaging, spectroscopy and imaging spectro

inverse problems for gamma-ray spectroscopy

spectrometer/telescope for imaging X-rays (STIX)

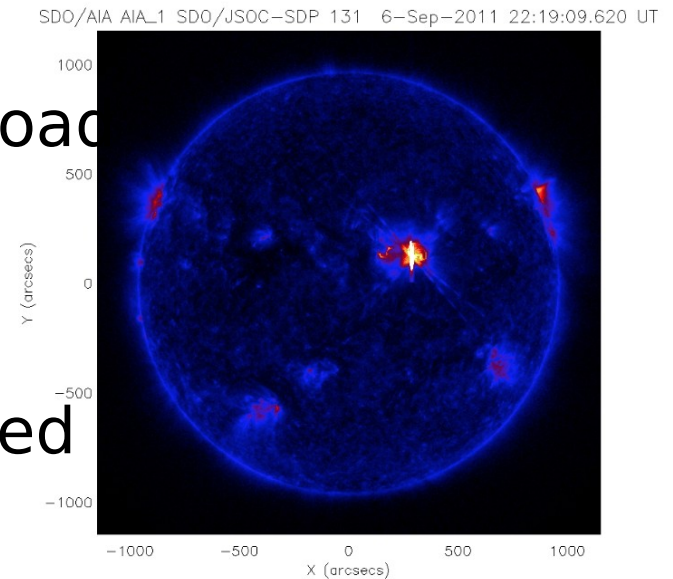
hard X-ray imaging spectroscopy
part of the solar orbiter payload (2019)
10 pairs of tungsten grids
10 cadmium-telluride detectors
mask pattern technique
spatial resolution: 1,4 km on the sun
energy range: 4 – 150 keV



use problems for imaging, spectroscopy and imaging spectro

atmospheric imaging assembly (AIA)

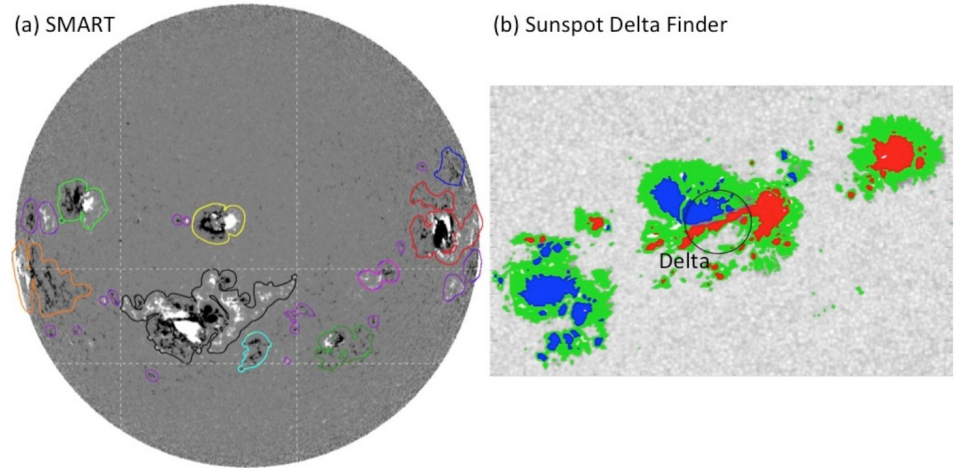
image de-saturation
launched on feb 2010 in the SDO payload
EUV wavelengths
4096x4096 images
spatial resolution: 0.6 – 1.5 arcsec
imaging technique: standard CCD-based
FOV: diffusion + diffraction



inverse diffraction for image de-saturation

helioseismic and magnetic imager (HMI)

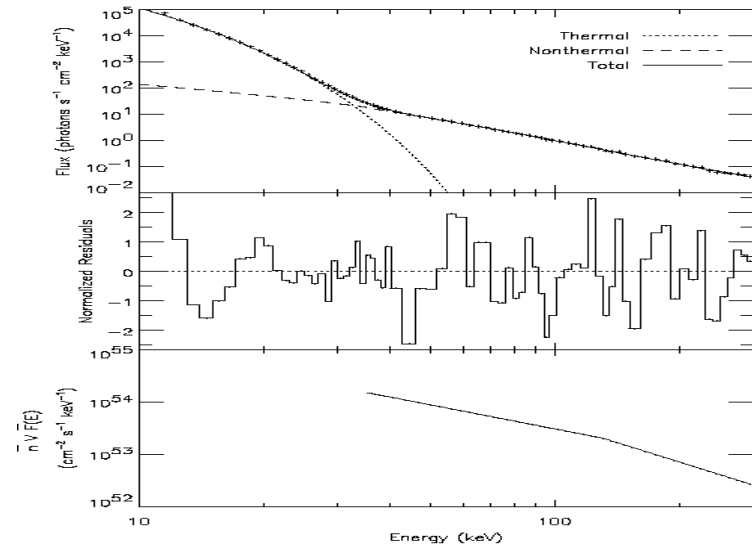
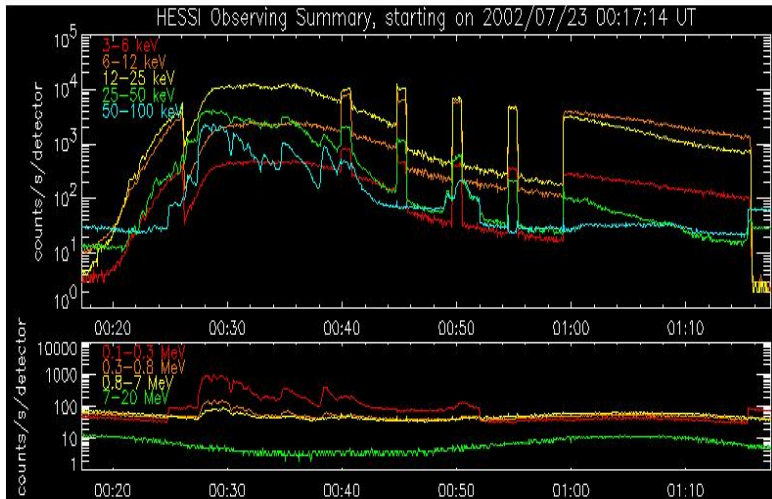
vector magnetograms
full disk images
time cadence: 12 minutes



flare prediction using machine learning

hard X-ray physics

spectroscopy



regularized inversion solar software (piana et al, ApJ, 2007)

given

$$I(\varepsilon) = \frac{nV}{4\pi R^2} \int_{\varepsilon}^{\infty} Q(\varepsilon, E) \bar{F}(E) dE$$

the software provides a reconstruction of

$$\bar{n} V \bar{F} = \int_V n(\vec{r}) F(E, \vec{r}) d\vec{r}$$

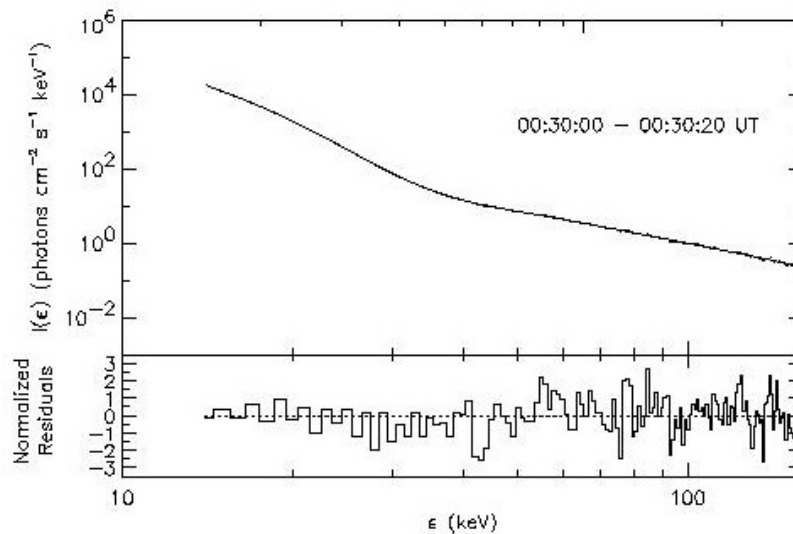
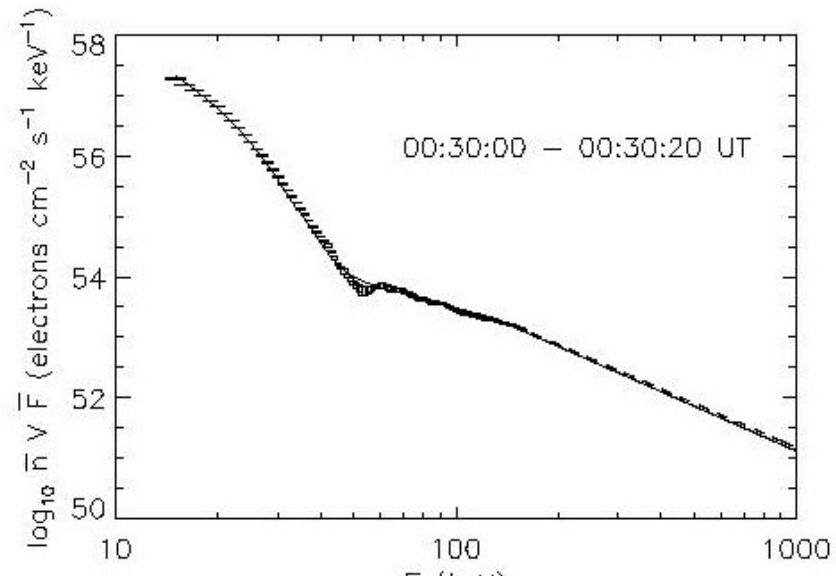
smoothed along E

- $Q(\varepsilon, E)$:
- all models up to 3BN in Koch and Motz (1959)
 - anisotropic effects (Massone et al 2004)
 - ee-bremsstrahlung effects (Kontar et al 2007)

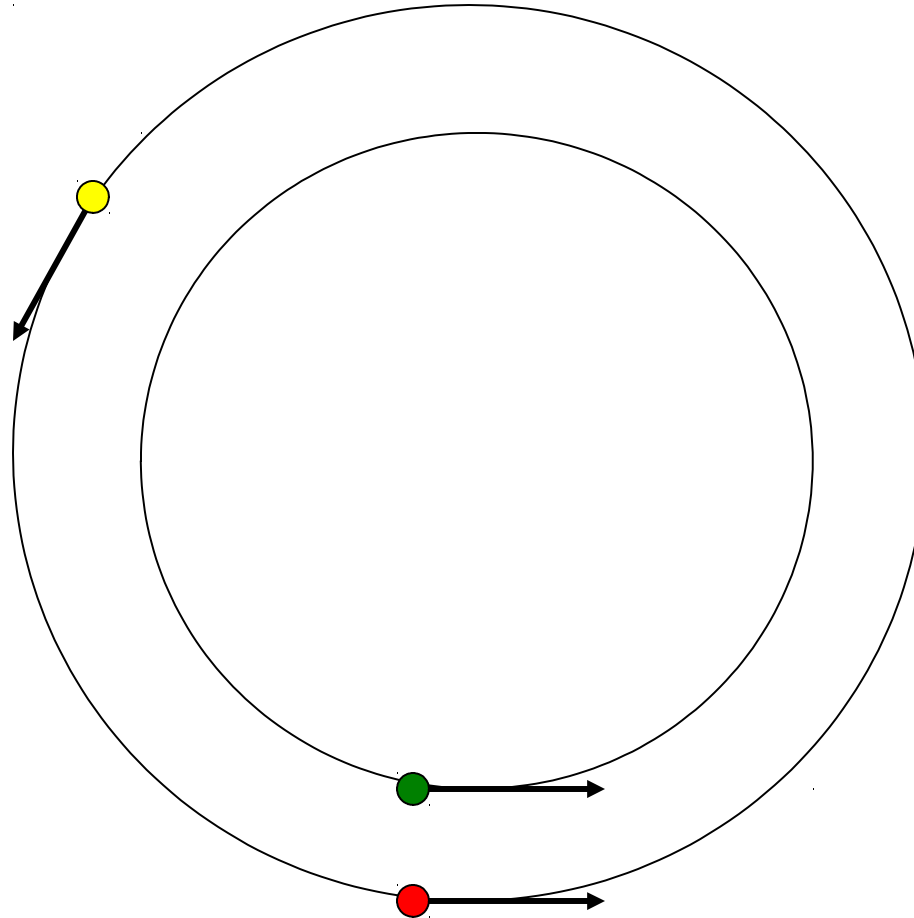
regularized inversion software

July 23 2002

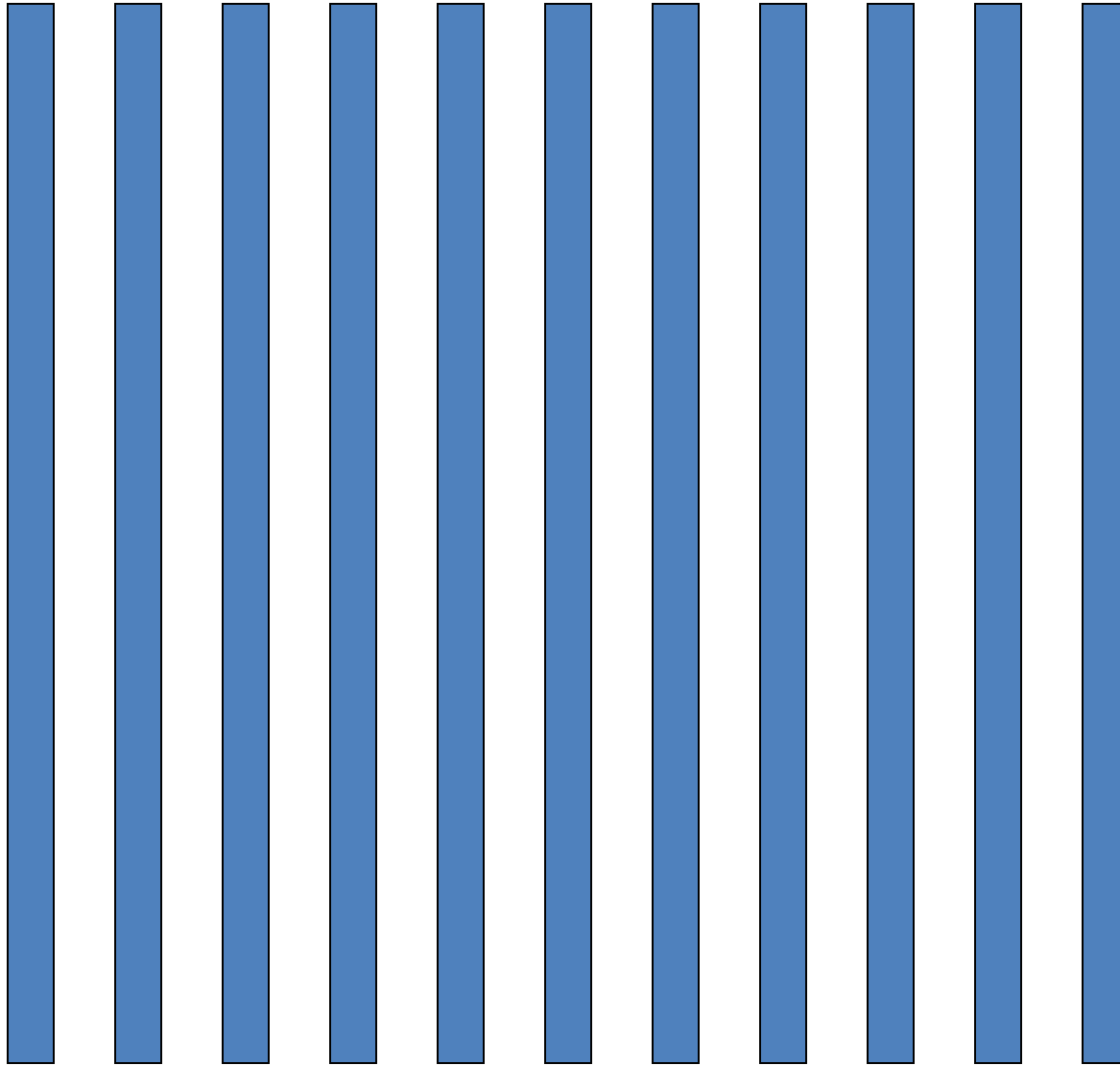
00:30:00 – 00:30:20 UT

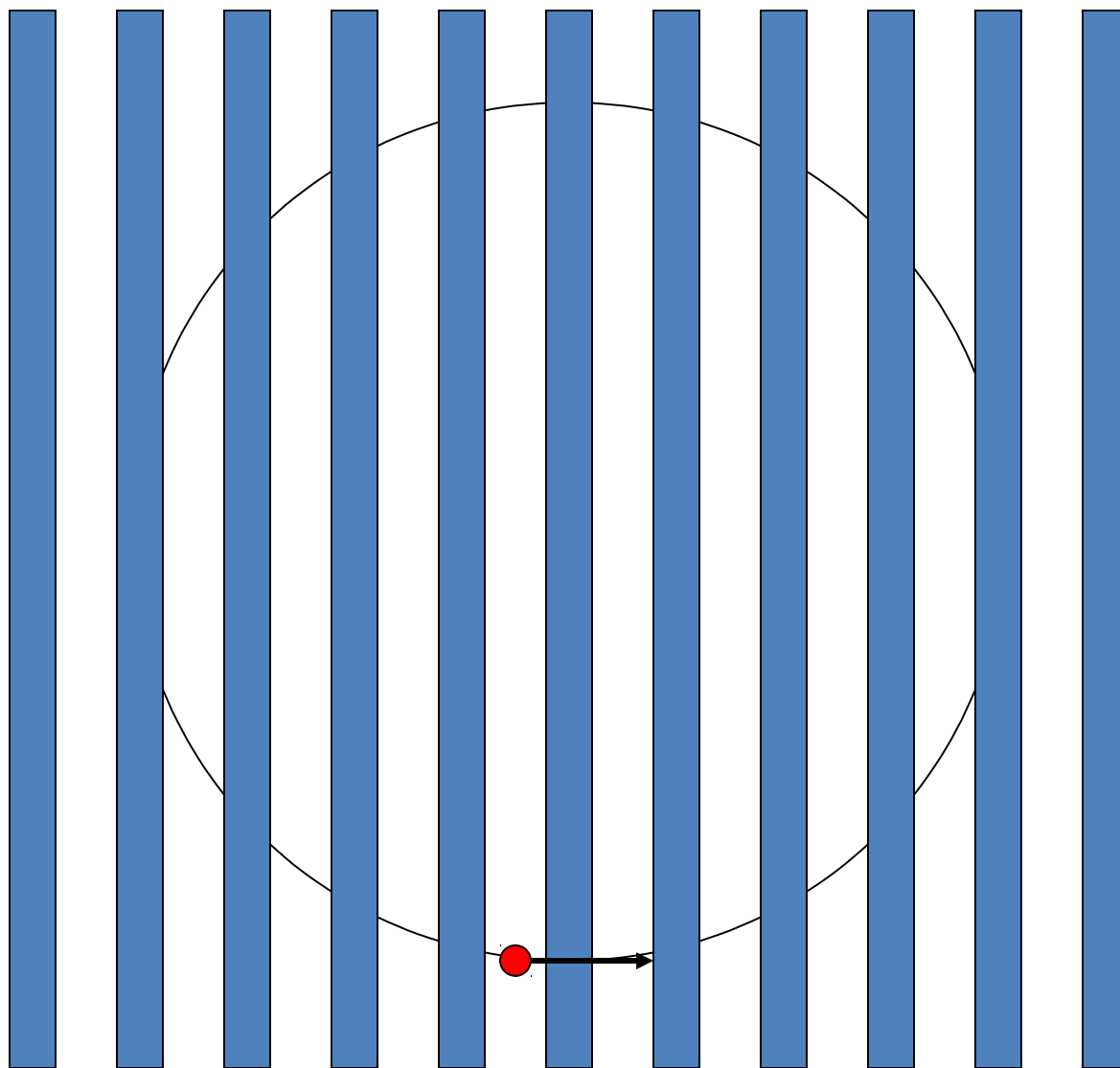


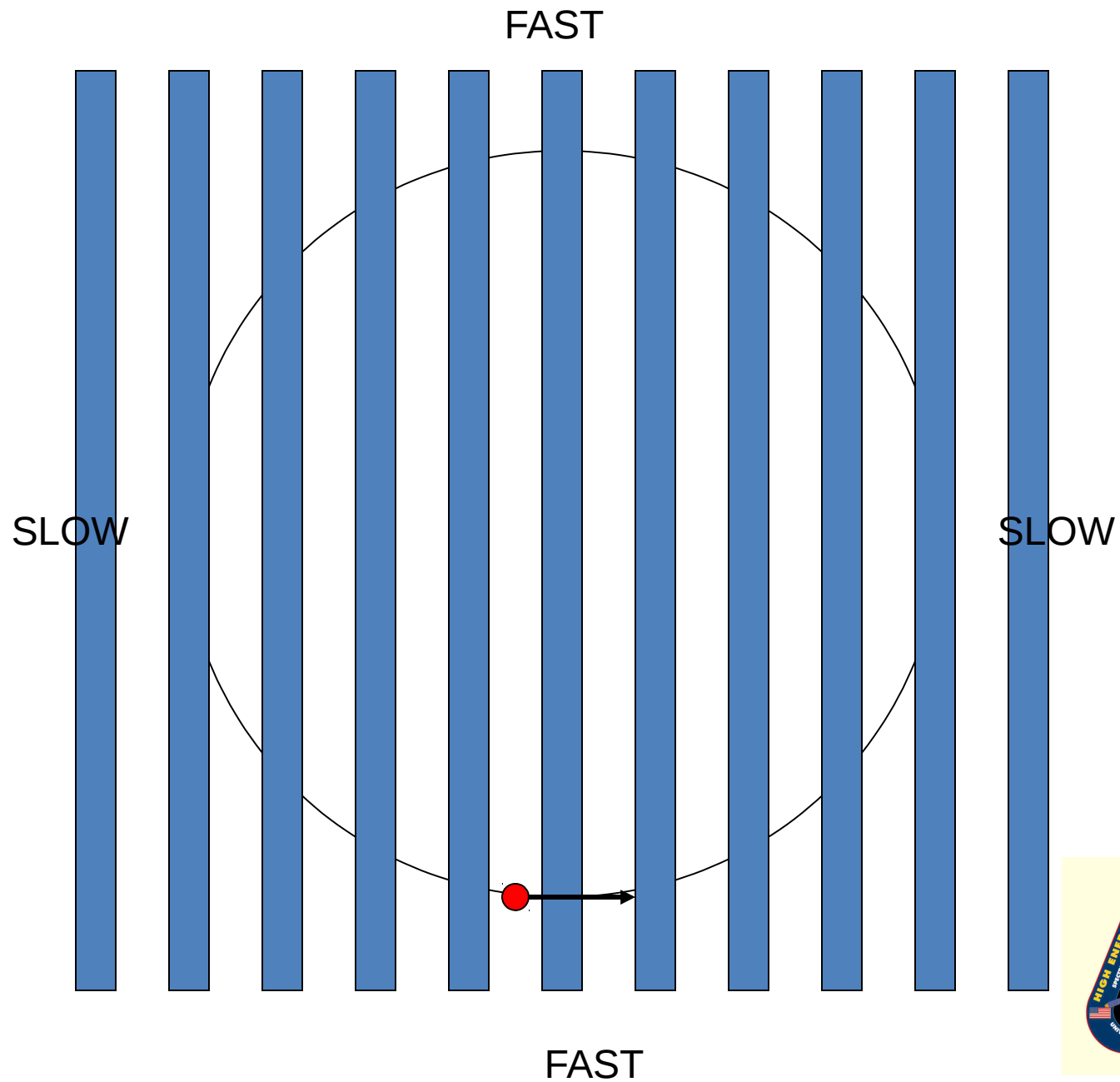
Rotation of point sources in spacecraft field of view

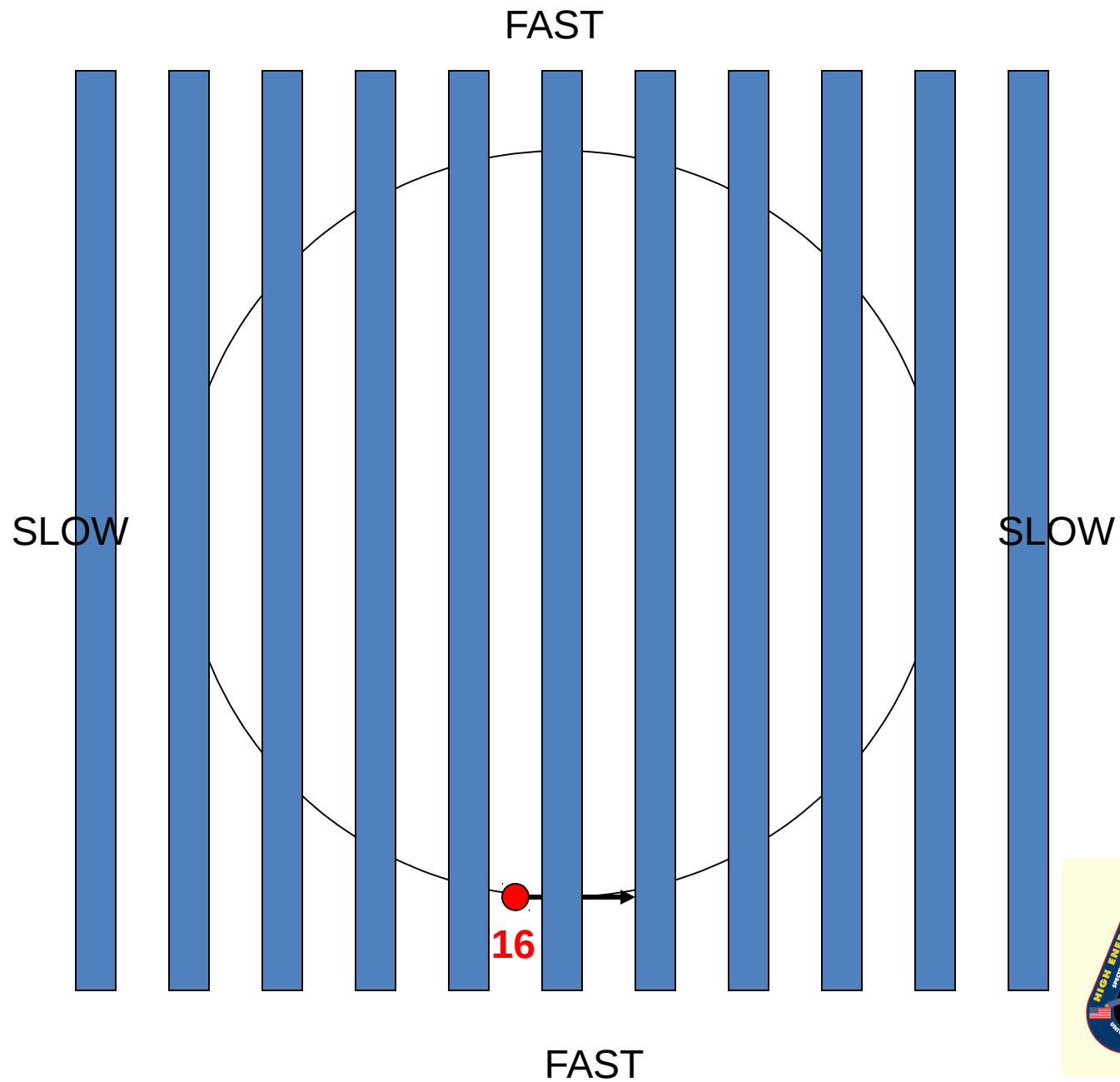


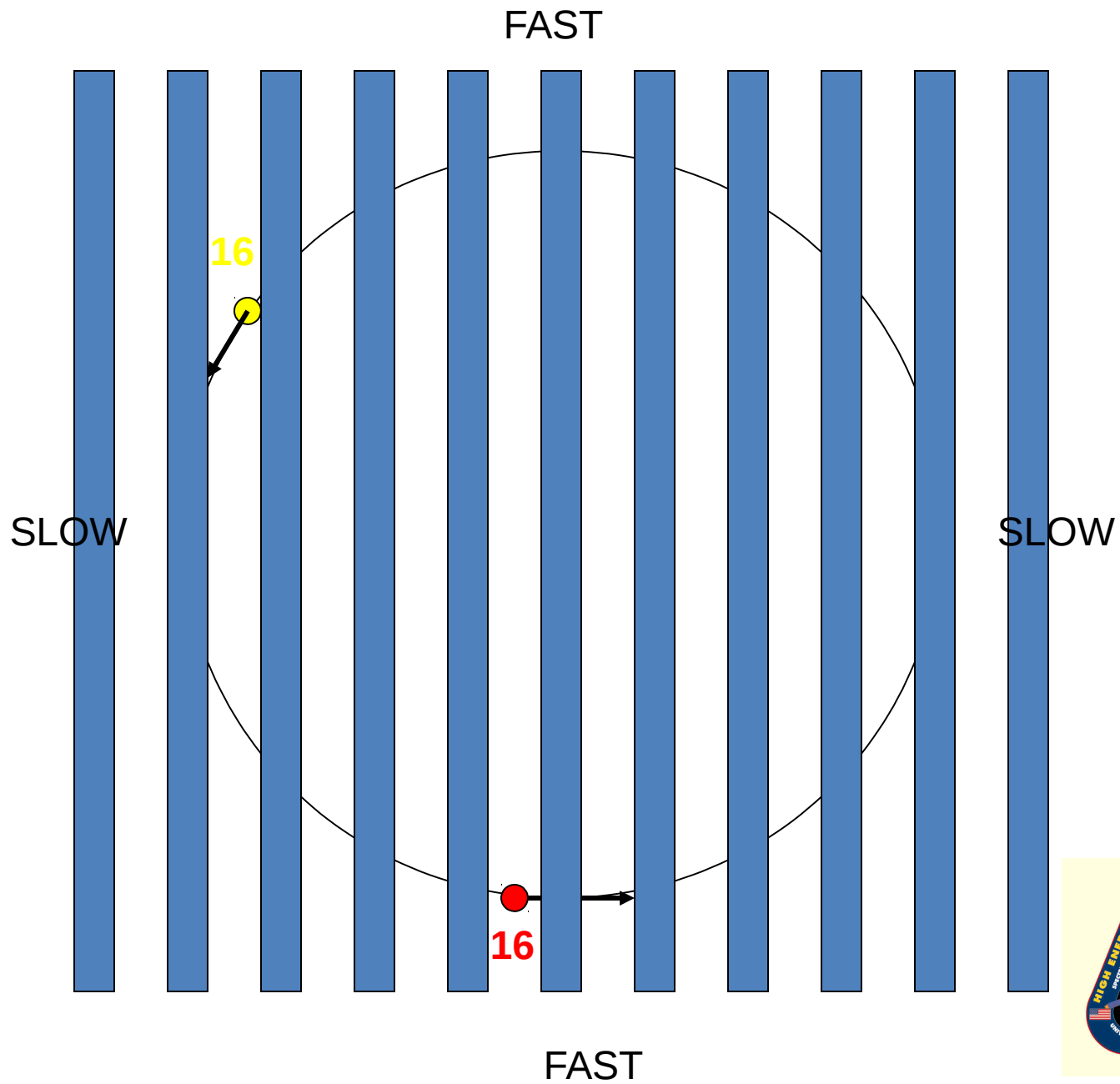
Grid Pattern

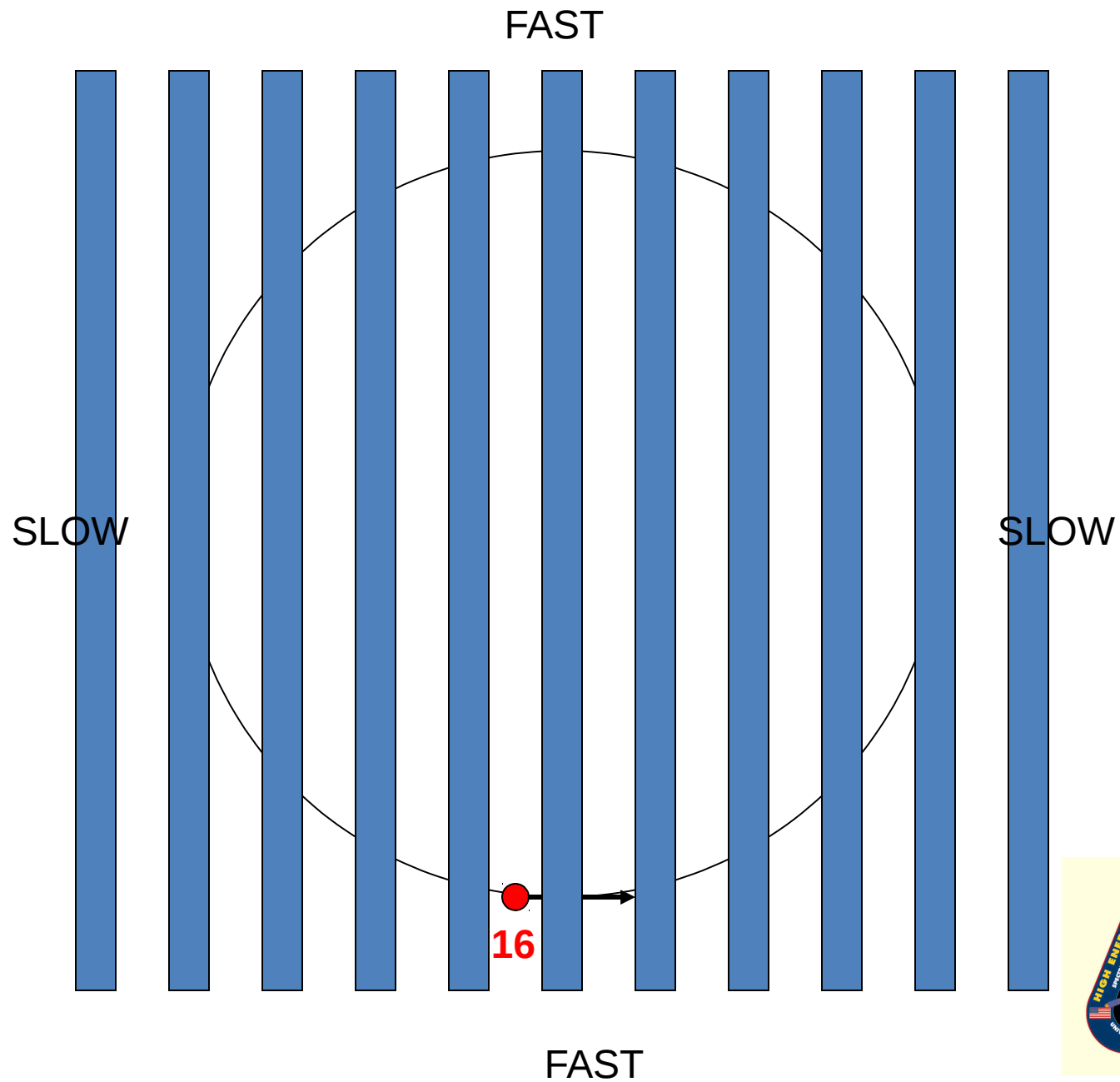


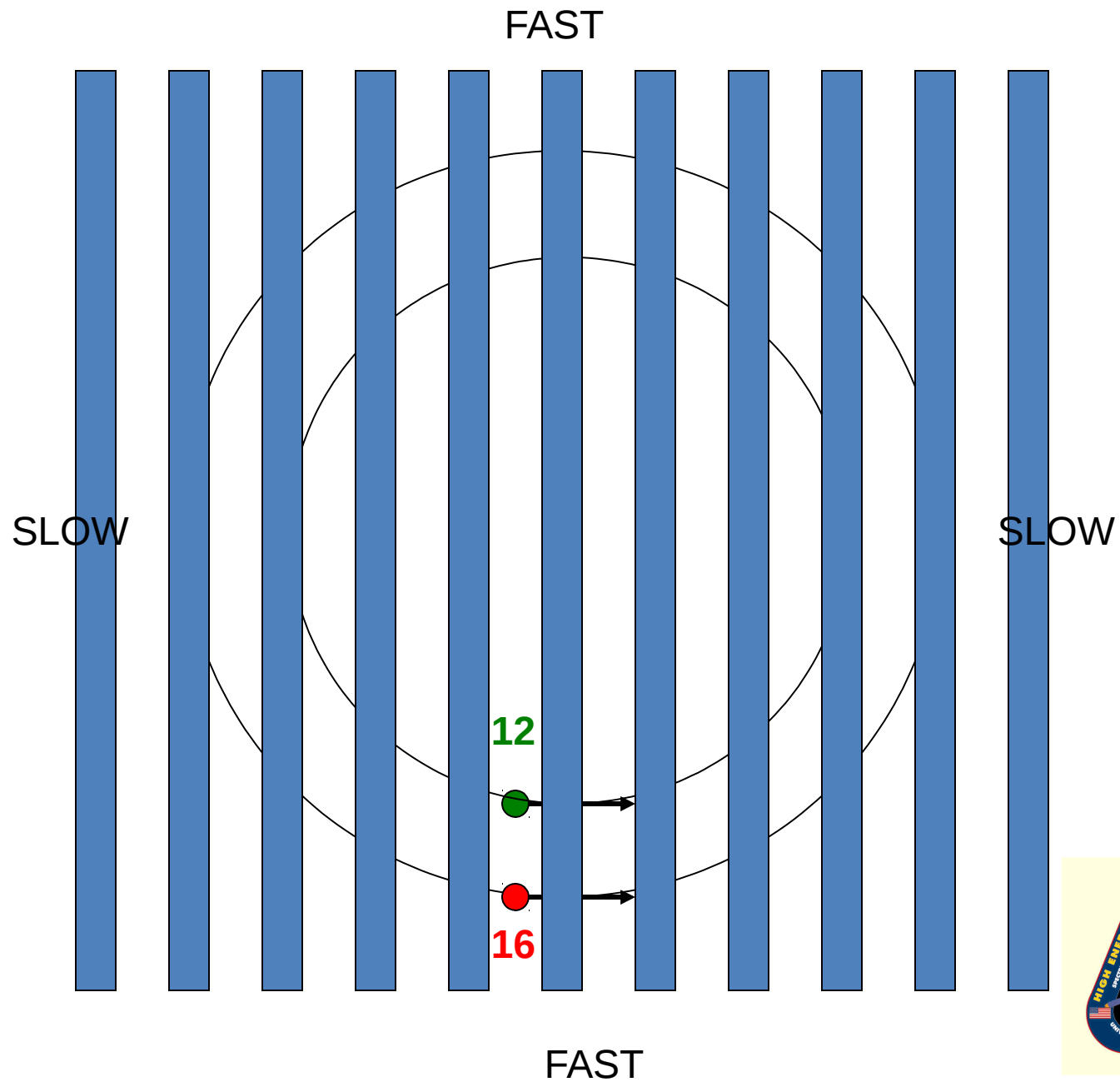




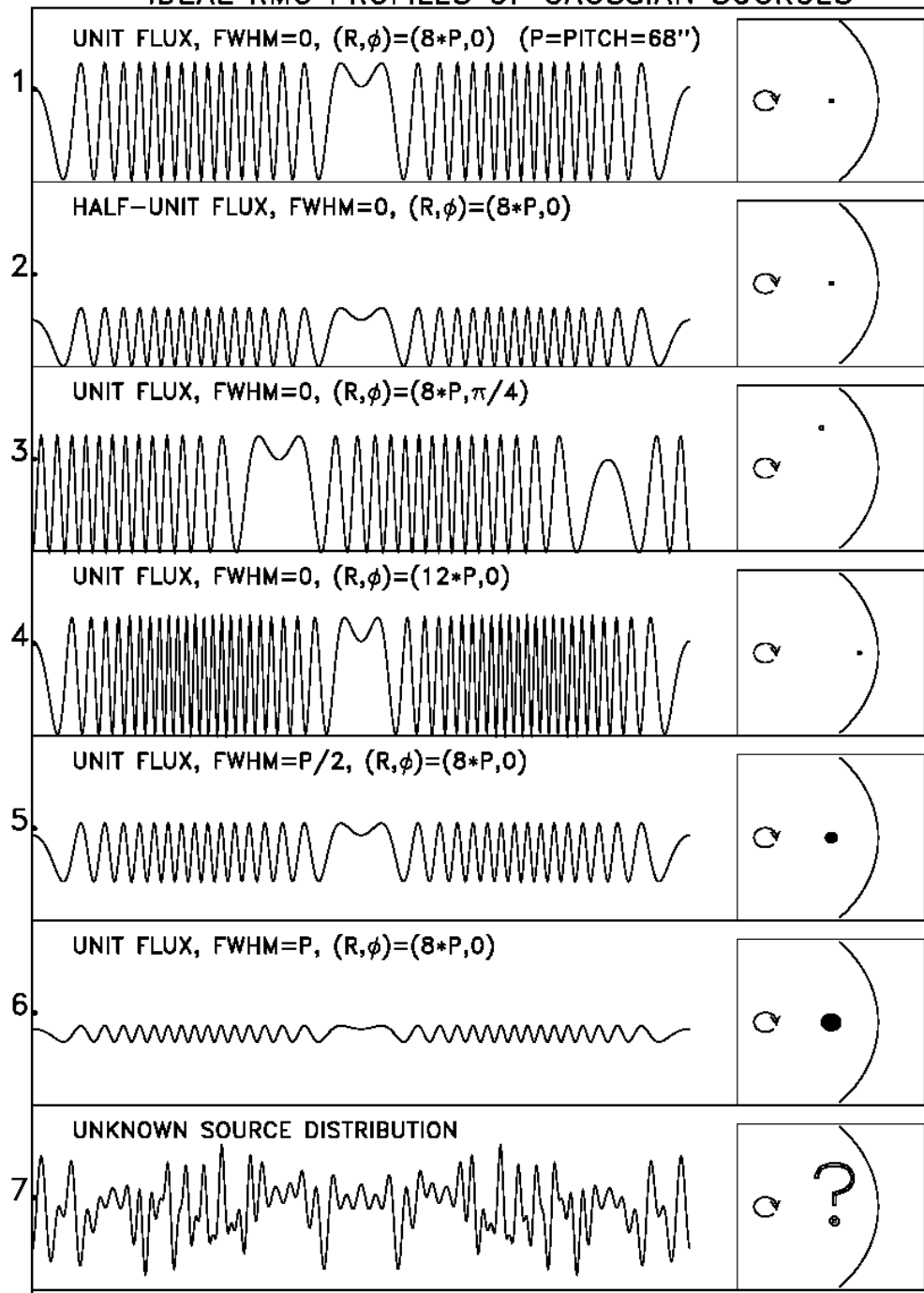








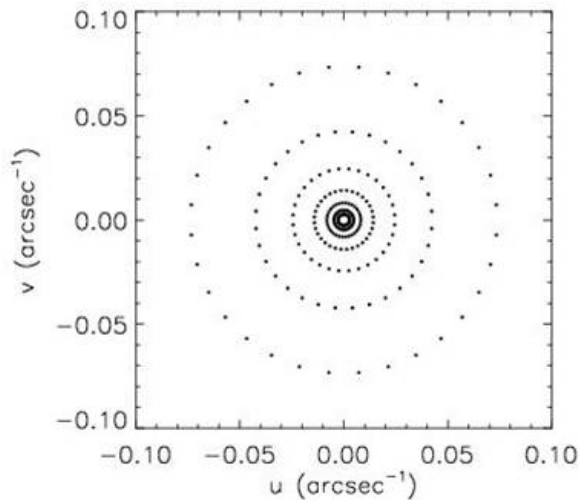
IDEAL RMC PROFILES OF GAUSSIAN SOURCES



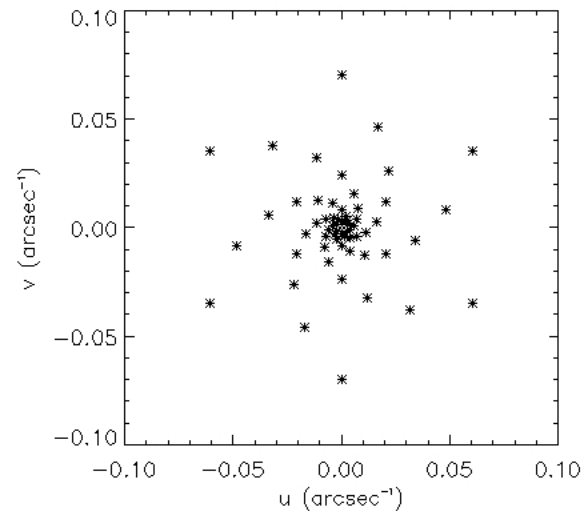
visibilities

rotating modulation collimators + data stacking = visibilities

$$V(u_j, v_j; \epsilon) = \int I(x, y; \epsilon) e^{-2\pi i(u_j x + v_j y)} dx dy$$



RHES SI



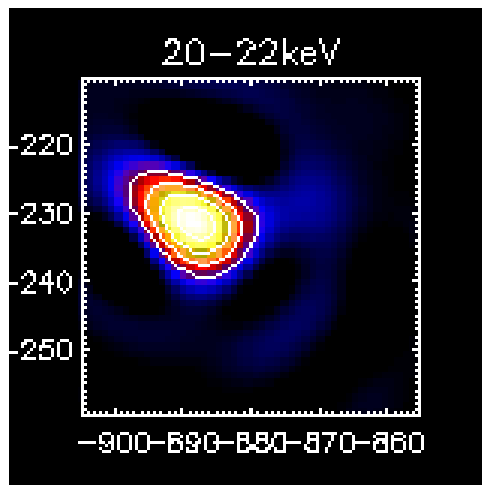
STIX

uv_smooth (massone et al, ApJ, 2009)

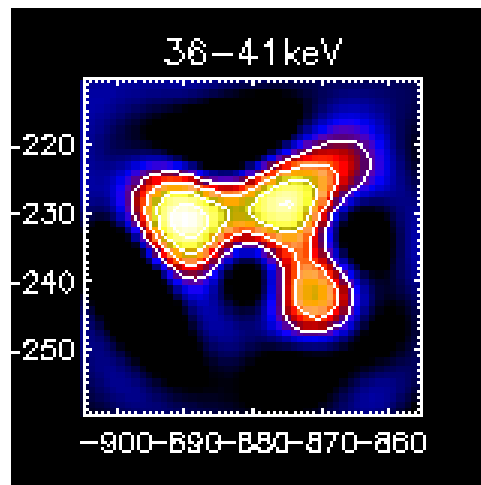
uv_smooth.pro in SSW:

1. interpolation of visibilities in the uv-plane
2. extrapolation by soft-thresholding

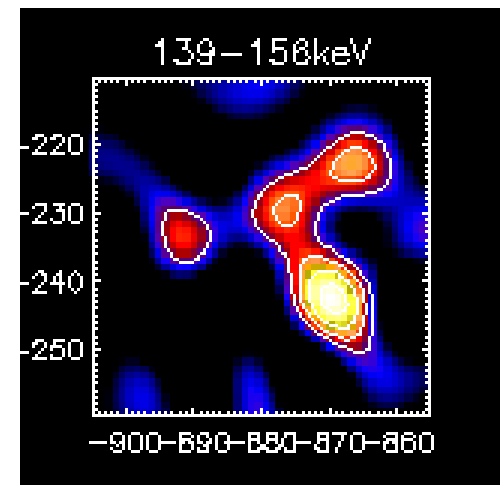
july 23 2002:



20-22 keV



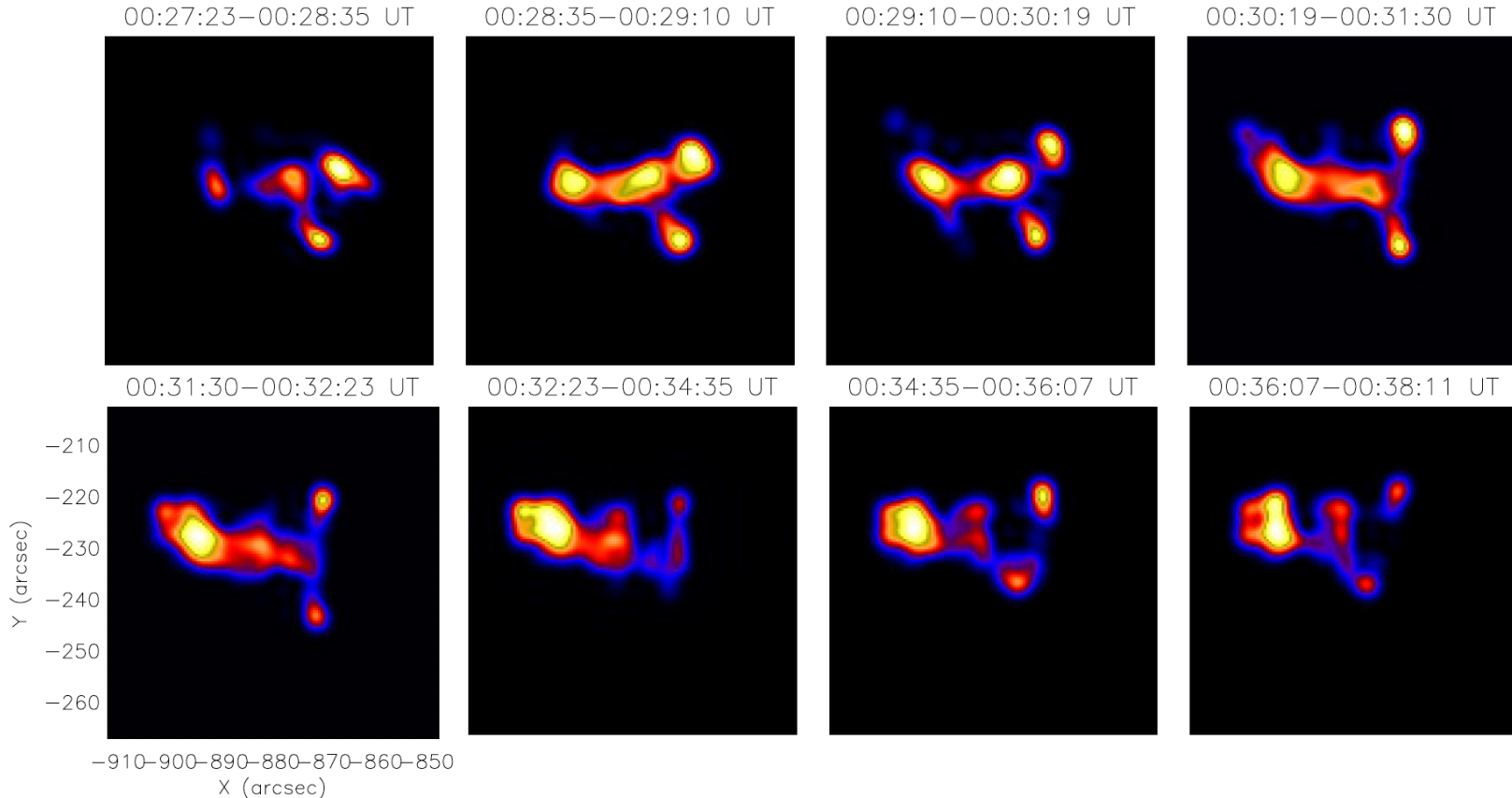
36-41 keV



139-156 keV

VIS_WV.pro in SSW (duval-poo, piana, massone, A&A, 2018)

$$\min_{\mathbf{f}} \left\{ \|\mathbf{H} \cdot \mathbf{F}\mathbf{f} - \mathbf{v}\|_2^2 + \lambda \|\mathbf{W}\mathbf{f}\|_1 \right\}$$



electron maps

(Piana et al ApJ 2007)

ously to count visibilities, electron visibilities can be defined

$$W(u, v; E) = \int \iint (x, y) \overline{F}(x, y; E) e^{2\pi i(ux+vy)} dx dy$$

t and electron visibilities are related by the equation:

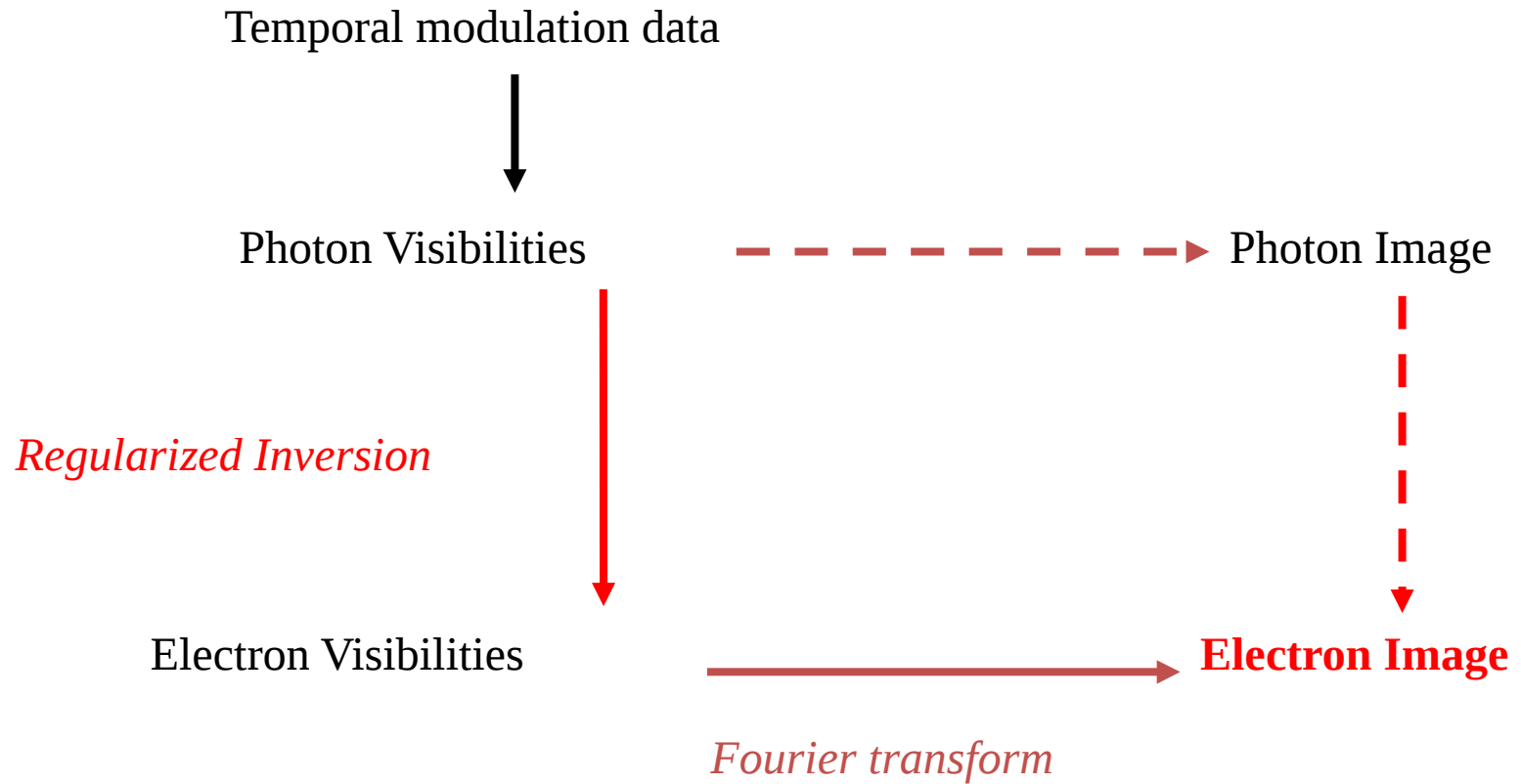
$$V(u, v; \varepsilon) = \frac{nV}{4\pi R^2} \int_{\varepsilon}^{\infty} Q(\varepsilon, E) W(u, v; E) dE$$

therefore:

ach (u,v), spectral inversion provides electron visibility spect

each E, image reconstruction provides an electron map

electron maps



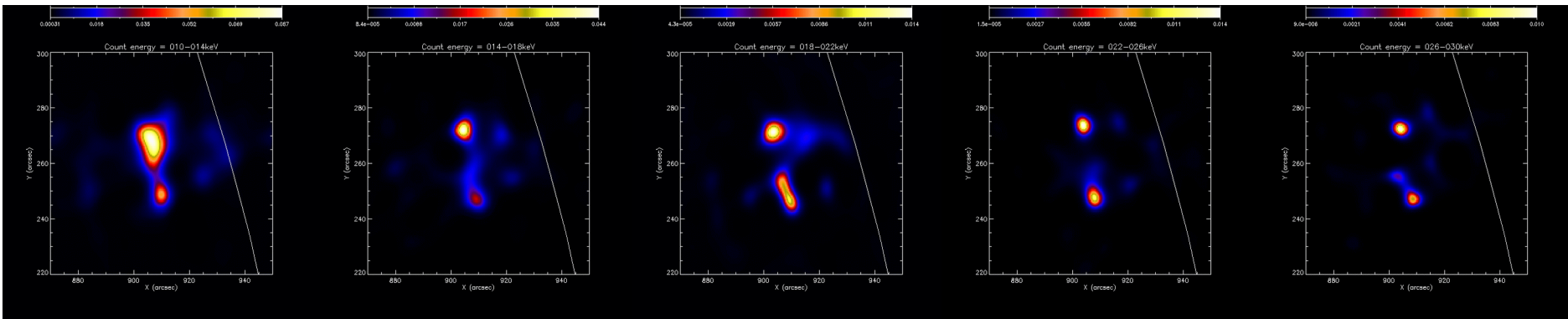
electron maps

current release:

input: count visibilities from most detectors at many count energies

output: electron visibilities at even more electron energies

electron maps (feburary 20 2002)



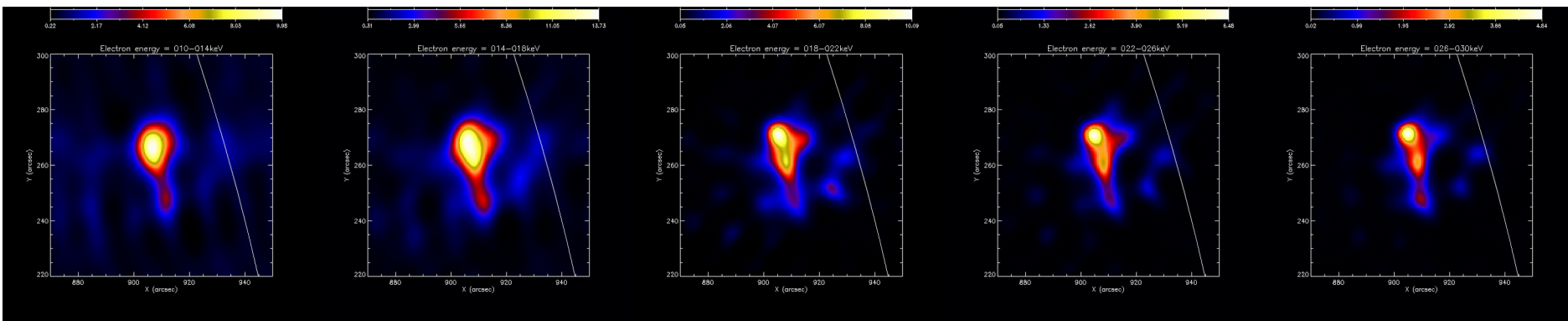
10-14
26

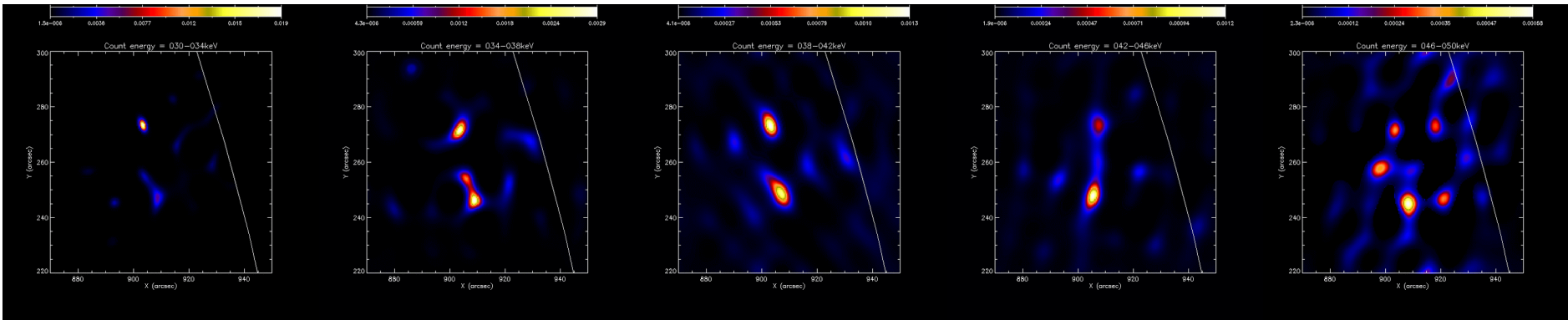
26-30

14-18

18-22

22-





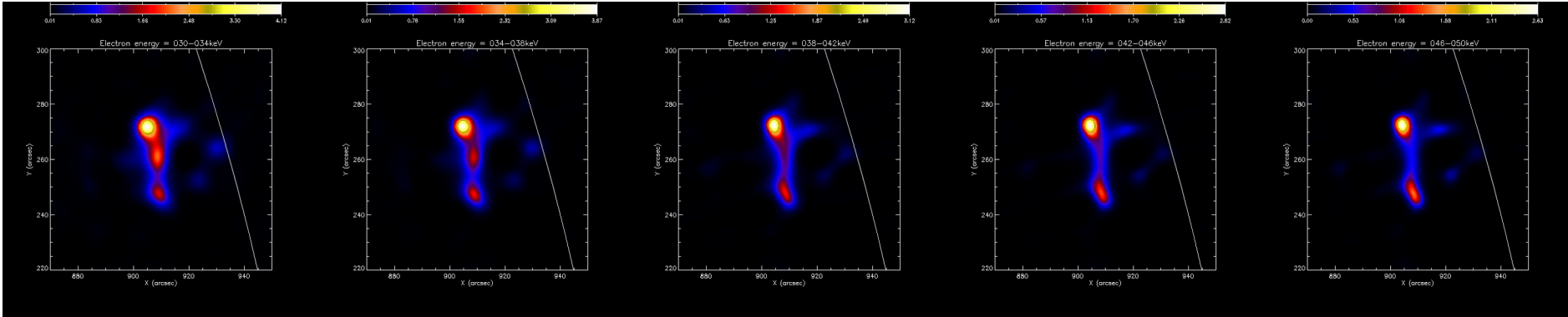
30-34
46

46-50

34-38

38-42

42-

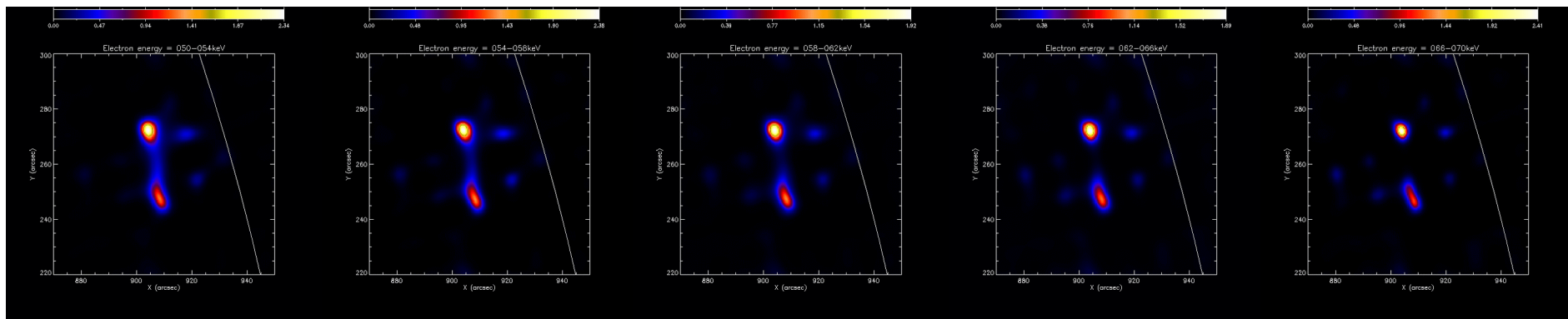


50-54
66

54-58
66-70

58-62

62-



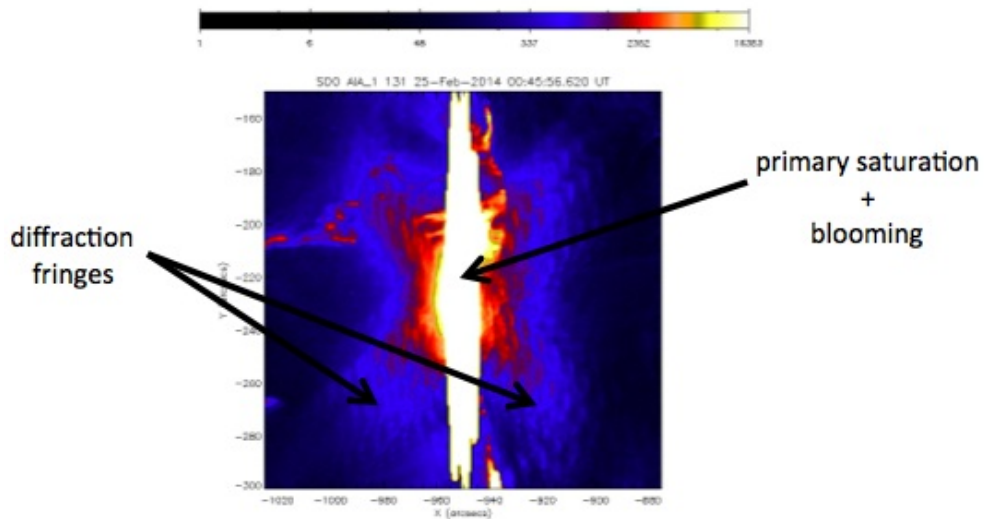
$$\frac{\partial F(E, s)}{\partial s} + \frac{\partial}{\partial E} \left(F(E, s) \frac{dE}{ds} \right) = S(E, s),$$

Table 2
Acceleration Region Characteristics

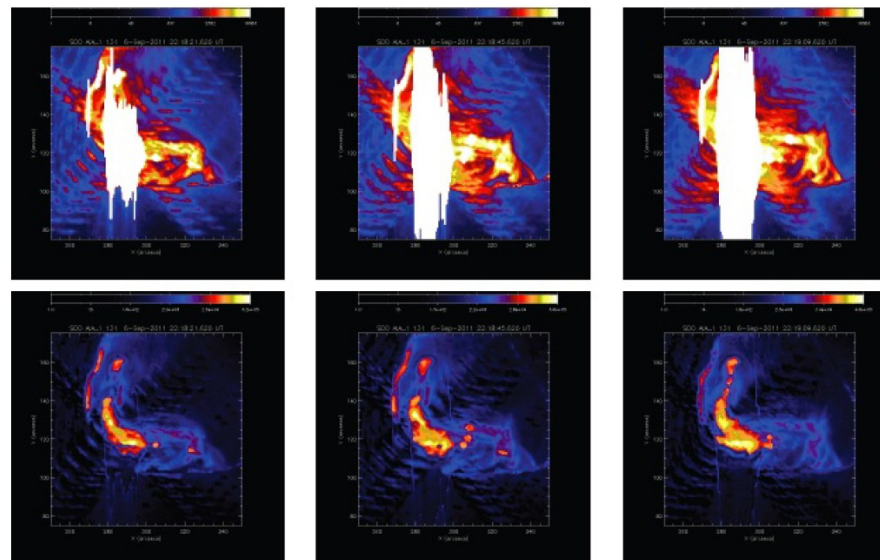
Event No.	L_0 (arcsec)	W (arcsec)	V_0 (100 arcsec ³)	n (10 ¹¹ cm ⁻³)	$\mathcal{N}$ (10 ³⁷)	η (20 keV) (10 ⁻³ s ⁻¹)	f
1	18.6	7.0	7.2	1.5	4.1	6.5	0.45
2	16.3	6.9	6.2	1.4	3.2	14.5	0.83
3	16.7	7.3	7.0	4.4	11.7	4.0	0.04
4	16.6	7.3	7.0	4.8	12.8	7.3	0.11
5	16.6	8.2	8.7	10.5	34.9	3.3	0.03
6	11.9	5.9	3.3	4.9	6.0	0.6	0.02
7	10.4	6.0	3.0	1.8	2.0	12.1	0.44
8	17.8	6.9	6.4	2.6	7.1	24.1	0.90
9	18.8	6.6	6.5	2.9	7.7	23.1	1.05
10	15.1	6.0	4.2	2.9	5.4	13.8	0.72
11	16.0	5.7	4.1	1.9	3.1	27.8	1.95
12	10.3	6.6	3.5	5.1	6.7	4.9	0.08
13	9.9	6.5	3.3	4.6	5.7	4.1	0.18
14	21.5	5.3	4.8	1.5	2.8	1.4	0.13
15	17.4	6.3	5.4	0.8	1.7	1.7	1.03
16	17.8	6.4	5.8	2.3	5.1	0.3	0.18
17	11.0	6.2	3.3	3.9	5.0	2.9	0.05
18	9.9	6.3	3.1	3.2	3.8	7.0	0.22
19	19.9	6.2	6.1	11.1	25.7	13.6	0.02
20	14.5	6.1	4.2	5.2	8.3	23.4	0.10
21	9.9	6.1	2.9	2.2	2.4	16.5	0.53
22	12.4	6.0	3.6	1.7	2.3	5.2	0.26
Geometric Mean	14.5	6.4	4.7	2.9	5.4	6.0	0.20
$\times/\div$	1.3	1.1	1.4	1.9	2.2	3.4	3.9

flare morphology (EUV)

de-saturation of EUV SDO/AIA images



inverse diffraction allows exploiting the information in the diffraction fringes to restore the information in the saturation region



flare prediction

FLARECAST



horizon 2020 service
for flare forecasting

analis georgoulis (academy of athens)

ct scientist: shaun bloomfield (northumbria university)

- algorithm leader: anna massone (UNIGE and CNR - SPIN, g

- technology leader: mp (UNIGE and CNR - SPIN, genova)

FLARECAST: the algorithms

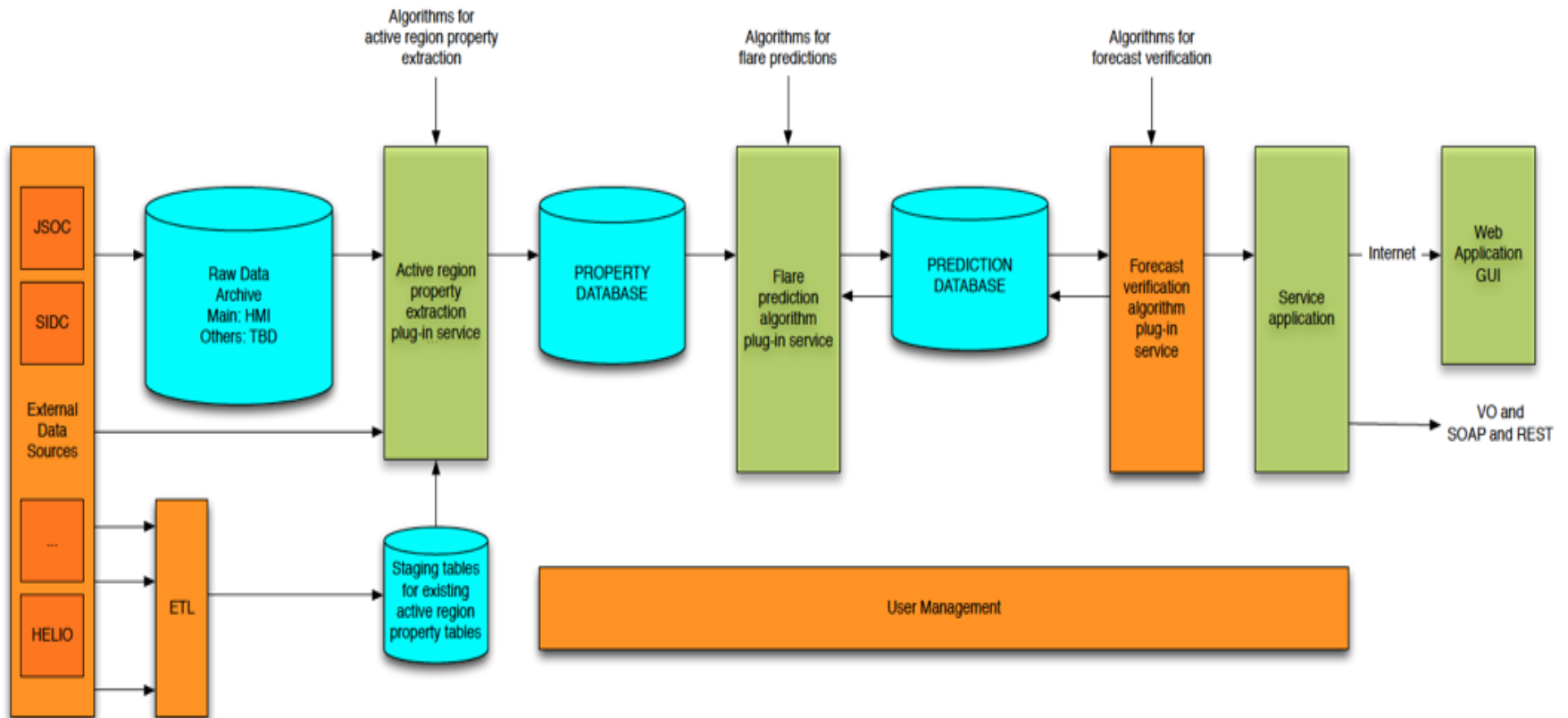
supervised methods

- LASSO
- hybrid LASSO
- elastic net
- logit
- hybrid logit
- random forest
- multi-layer perceptron
- recurrent neural network
- support vector machine
- garson method
- olden method

unsupervised methods

- K-means
- fuzzy C-means
- possibilistic C-means
- simulated annealing

FLARECAST: the platform



feature ranking: results - 1

	at least C1 flares						
	Hybrid Lasso	Hybrid Logit	SVC	Random Forest		average	std
flare_index_past	13,98	28,84	19,91	3,51		16,56	10,63
sharp_kw/hgradbh/total	3,47	37	18,59	16,57		18,95	13,87
wlsg_br/value_int	3,74	14,43	22,86	43,14		21,04	16,68
sharp_kw/jz/max	26,05	28	16,94	18,58		22,27	5,29
sharp_kw/usiz/max	24,2	36,75	34,79	18,37		28,53	8,73
wlsg_blos/value_int	3,96	45	46,61	25,39		30,24	20,00
r_value_br_logr	3,52	2,81	128,91	7,47		35,68	62,19
sharp_kw/ggt45fract	14,99	32	57,6	49,3		38,35	19,00
sharp_kw/usiz/stddev	17,15	49,46	54,26	45,89		41,69	16,72
sharp_kw/gamma/total	61,65	20,76	52,95	34,67		42,51	18,35

Lagrangian Coherent structures in magnetized plasmas

Daniela Grasso

Istituto Sistemi Complessi – CNR and Politecnico di Torino

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1. Istituto Gas Ionizzati – CNR

2. Università Roma Tre

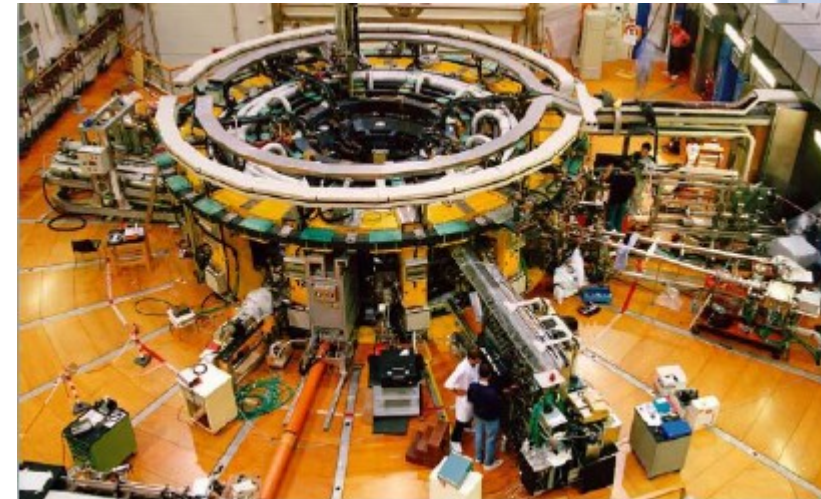
3. Università di Pisa

4. Eindhoven University of Technology

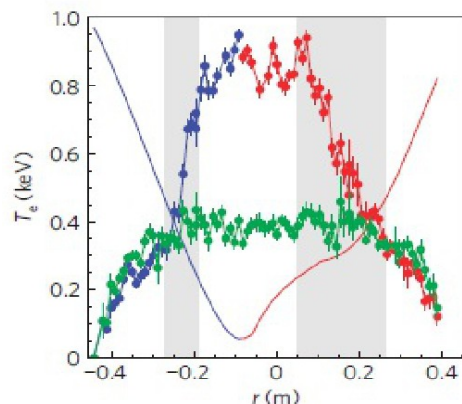
Motivation

Increasing the plasma current RFX-mod experiments show (RFX group, Nature 2009):

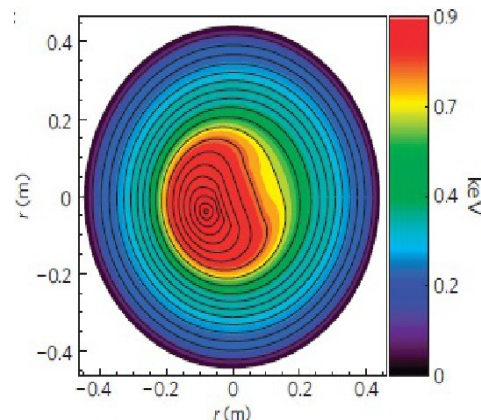
- transition from chaotic configurations to ordered ones, named Quasi Single Helicity (QSH) states;
- consequent formation of a strong transport barrier enclosing high temperature zone.



Electrons temp. vs r



2D map electrons Temp.

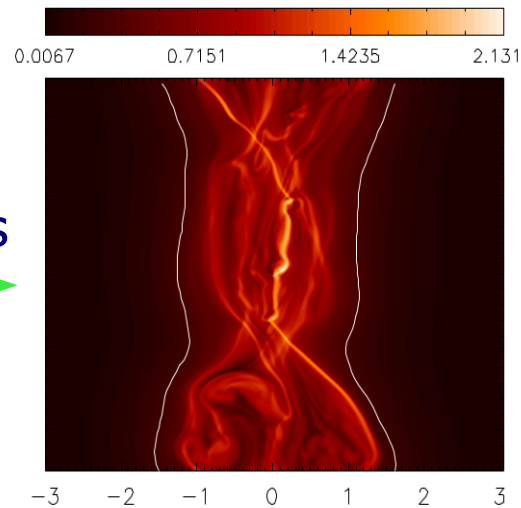


Does exist a correlation between the transport and the magnetic barriers?

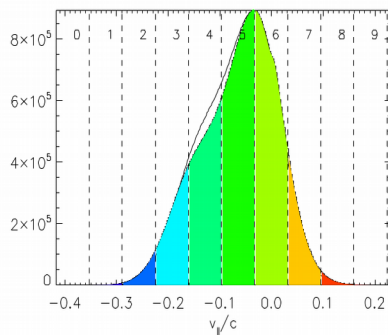
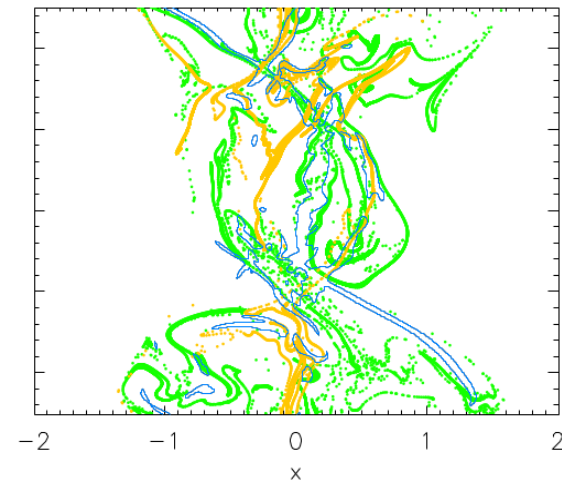
Motivation

- Test electron simulations in a chaotic magnetic field show a link between electron distribution and coherent structures (Borgogno, Perona, Grasso PoP 24, 122303(2017)).

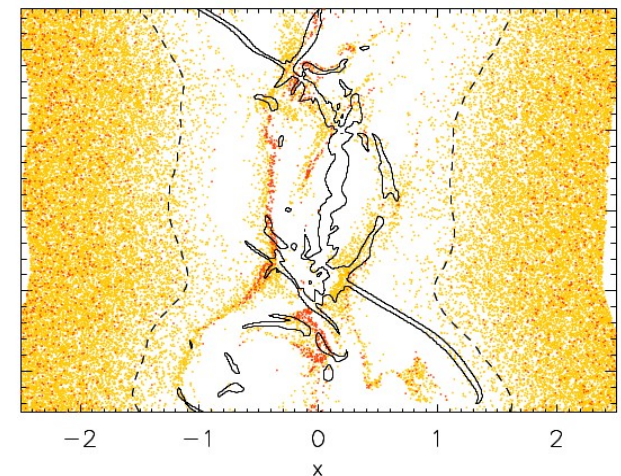
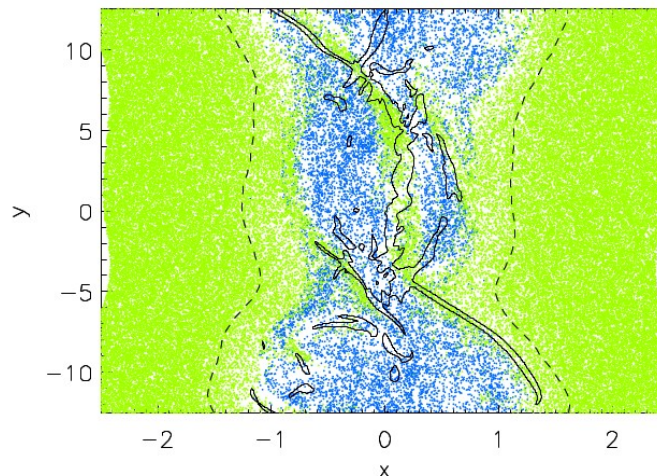
Current sheets



Manifolds



Electron
distribution



The problem of transport in a chaotic magnetic field

- In a magnetic chaotic field regions characterized by different behavior of the field line motion may coexist.
- How to relate the particle transport to the field line motion is an open problem.
- Here we present our approach based on the concept of Lagrangian Coherent Structures (LCS).
- We consider a model of chaos, derived from the results of a numerical simulation of a magnetic reconnection event, in which we assume electrons move along magnetic field lines.
- Two methods, ridges and most repelling/attracting material lines, borrowed from oceanography studies, have been successively adopted to find the LCS.

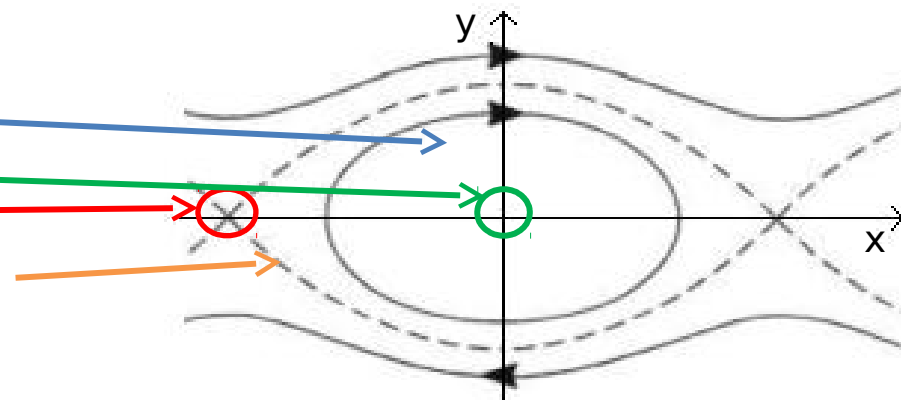
Chaotic magnetic field generated by a reconnection event

- Magnetic reconnection is a process occurring in conducting fluids and plasmas.
- It is a topological variation of the magnetic field, accompanied by a fast release of magnetic energy into heat and ordered kinetic energy.
- It is a local process which causes global changes, typically accompanied by the formation of intense current density layers.

2D geometry

- Magnetic island
- Fixed points: O-point
- X-point
- Separatrix

Magnetic field topology



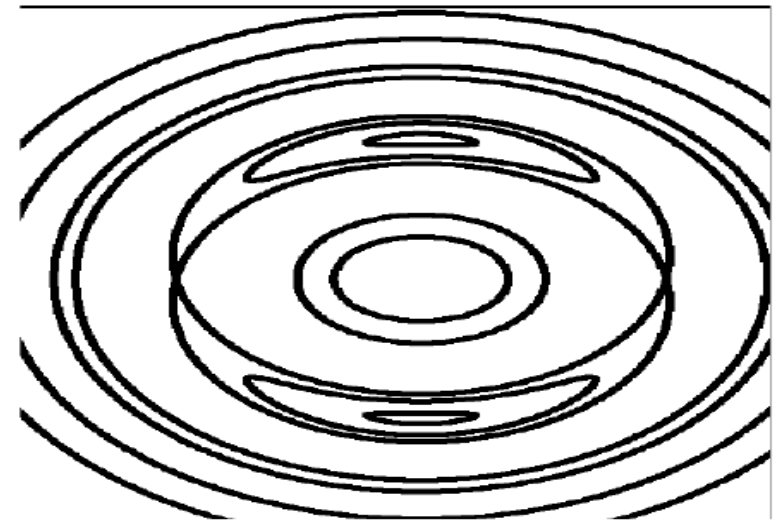
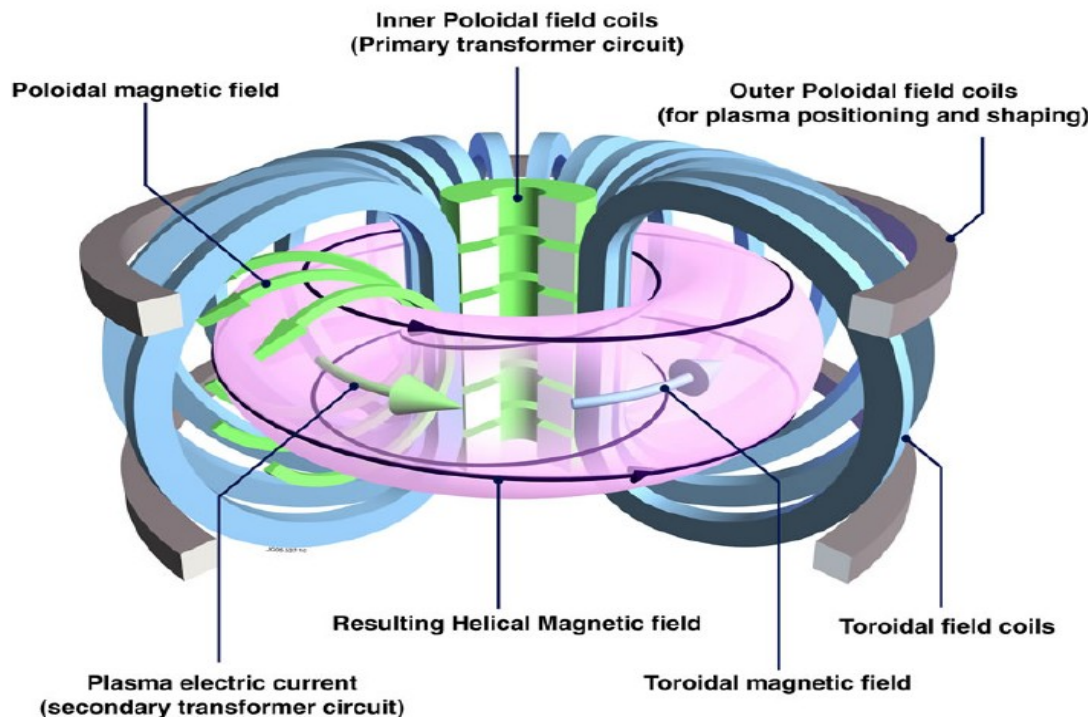
Chaotic magnetic field generated by a reconnection event

3D geometry

B_{tor} = toroidal magnetic field
 B_{pol} = poloidal magnetic field

Magnetic field lines wind helically on magnetic surfaces

Field line winding number
 $q = r B_{\text{tor}} / R b_{\text{pol}}$



- Magnetic reconnection occurs at resonant surfaces characterized by rational values of $q=m/n$, where m and n are the poloidal and toroidal wave numbers
- Magnetic reconnection is seen as a chain of island in cross section

Chaotic magnetic field generated by a reconnection event

- In 3D configurations there are several resonant surfaces



Island chains with different helicities

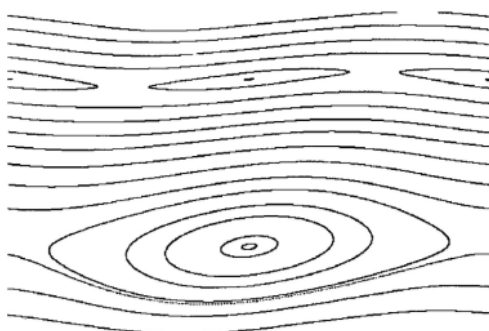
- Interaction between islands with different helicities



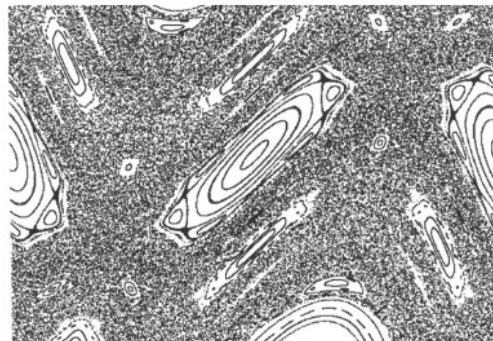
Magnetic chaos

- More restrictive case: two close resonant surfaces with $q_1=m/n$ and $q_2=m'/n'$ such that $m'n - n'm = \pm 1$.

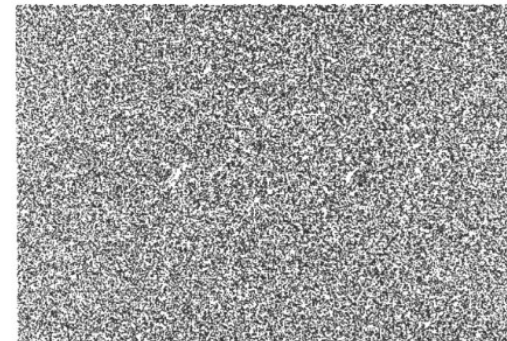
- Overlap criterion: $s = \frac{W_1 + W_2}{2\Delta x_{12}} > 1$, W island amplitude, Δx_{12} surfaces distance.



$s < 1$ single helicity approximation

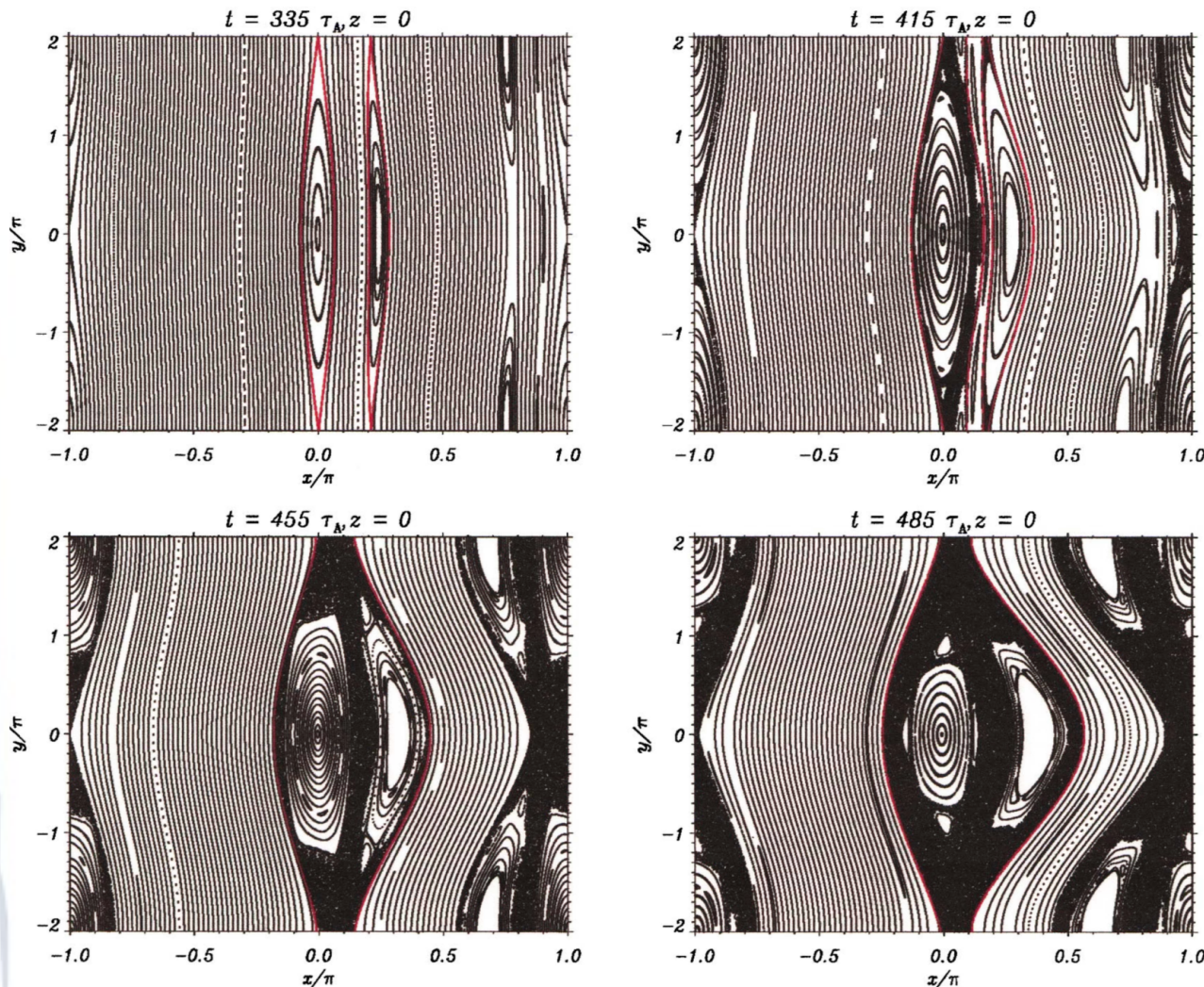


$s \sim 1$ chaos development



$s \gg 1$ completely chaotic domain

3D model for reconnection



- A reconnection event has been induced by a double helical perturbation resonating at two different surfaces.
- As shown in figure, the islands size increase until they start to influence each other leading to a chaotic setting initially enclosed between the islands
- Do some barriers exist influencing the transport in this chaotic sea?

3D model for reconnection

- Magnetic field:

$$\vec{B} = B_0 \vec{e}_z + \nabla \psi(\vec{x}, t) \times \vec{e}_z$$

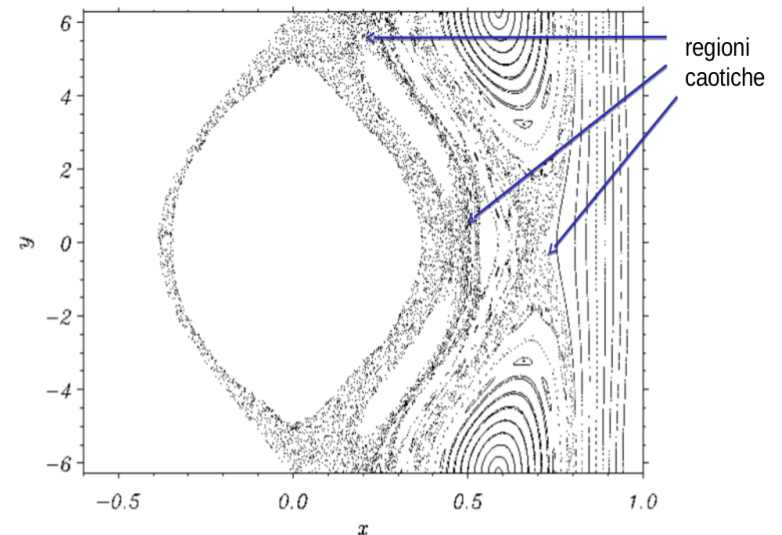
- Magnetic field line equations:

$$\frac{dx}{dz} = -\frac{\partial \psi}{\partial y}; \quad \frac{dy}{dz} = \frac{\partial \psi}{\partial x}$$

for fixed time, t , z plays the role of field line time.

- Poincaré plots give a detailed picture of the chaotic domain of the magnetic field, provided one integrates the Hamilton equations starting from a sufficiently large number of i.c. and for a sufficiently long time.

NO INFORMATION ON TRANSPORT!

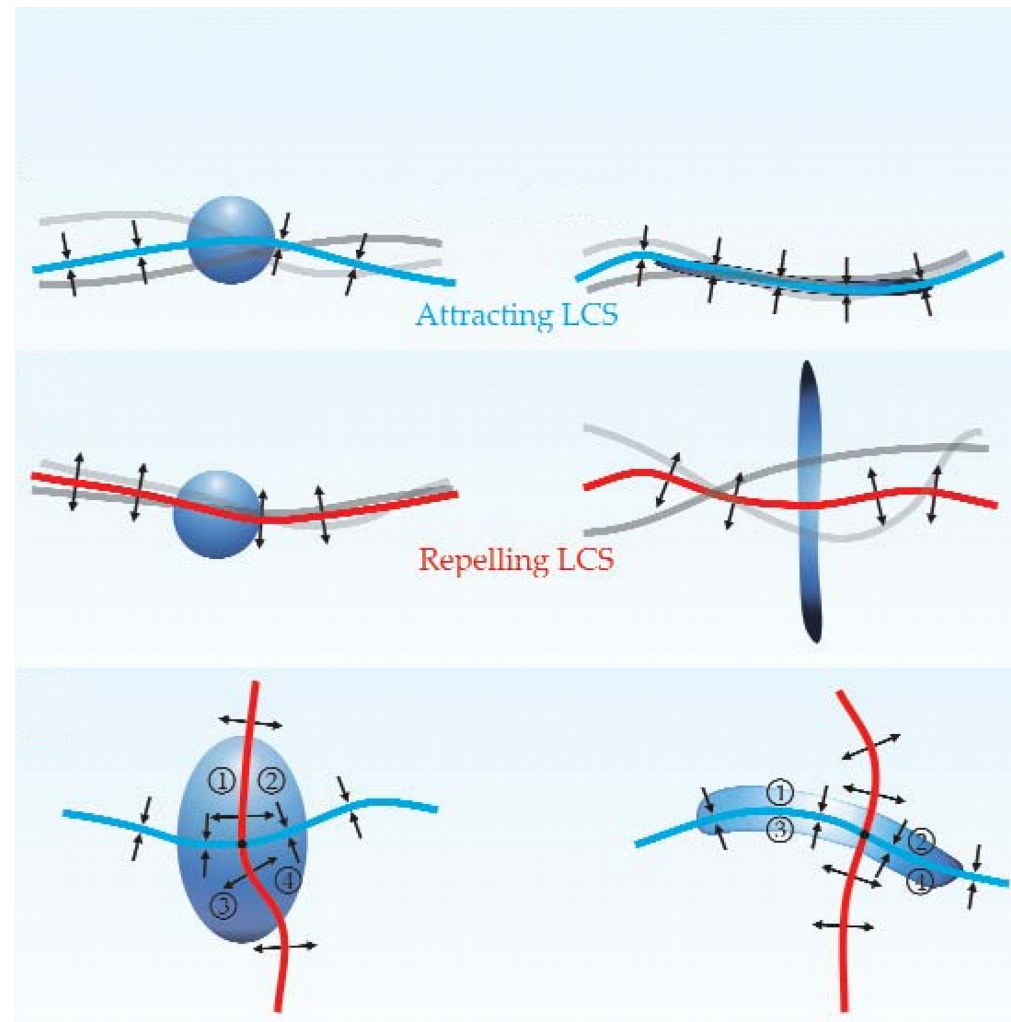


How to identify barriers between different chaotic regions?

- We define the barriers that may form in the chaotic sea as Lagrangian Coherent Structures.
- LCS organize the flows in macroregions whose fluid elements may not exchange and hence act as transport barriers over a finite time interval.
- How to identify these barriers is an open issue and several methods have been proposed.
- We successfully adopted the ridges of the Finite Time Lyapunov Exponent Field based on Haller 2001; Shadden 2005-2007 in the past (see Borgogno et al. PoP 18, 102307 (2011); Rubino et al. 57 085004 (2015); Veranda et al. 57, 116029 (2017)).
- Here we adopt the more refined definition successively proposed by Haller 2010, 2014 based on the most repelling/attracting material lines.

Most repelling/attracting material lines

Graphic definition:



Most repelling material lines

Mathematical definition: A LCS over a finite time interval Z is defined as a material line satisfying in each point the following conditions on the eigenvalues and eigenvectors of the Cauchy Green strain tensor M :

i) $\lambda_{min} < \lambda_{max} ; \lambda_{max} > 1$

ii) $e_0 = \xi_{min}$

the tangent vector is along the eigenvector associated with the smallest eigenvalue

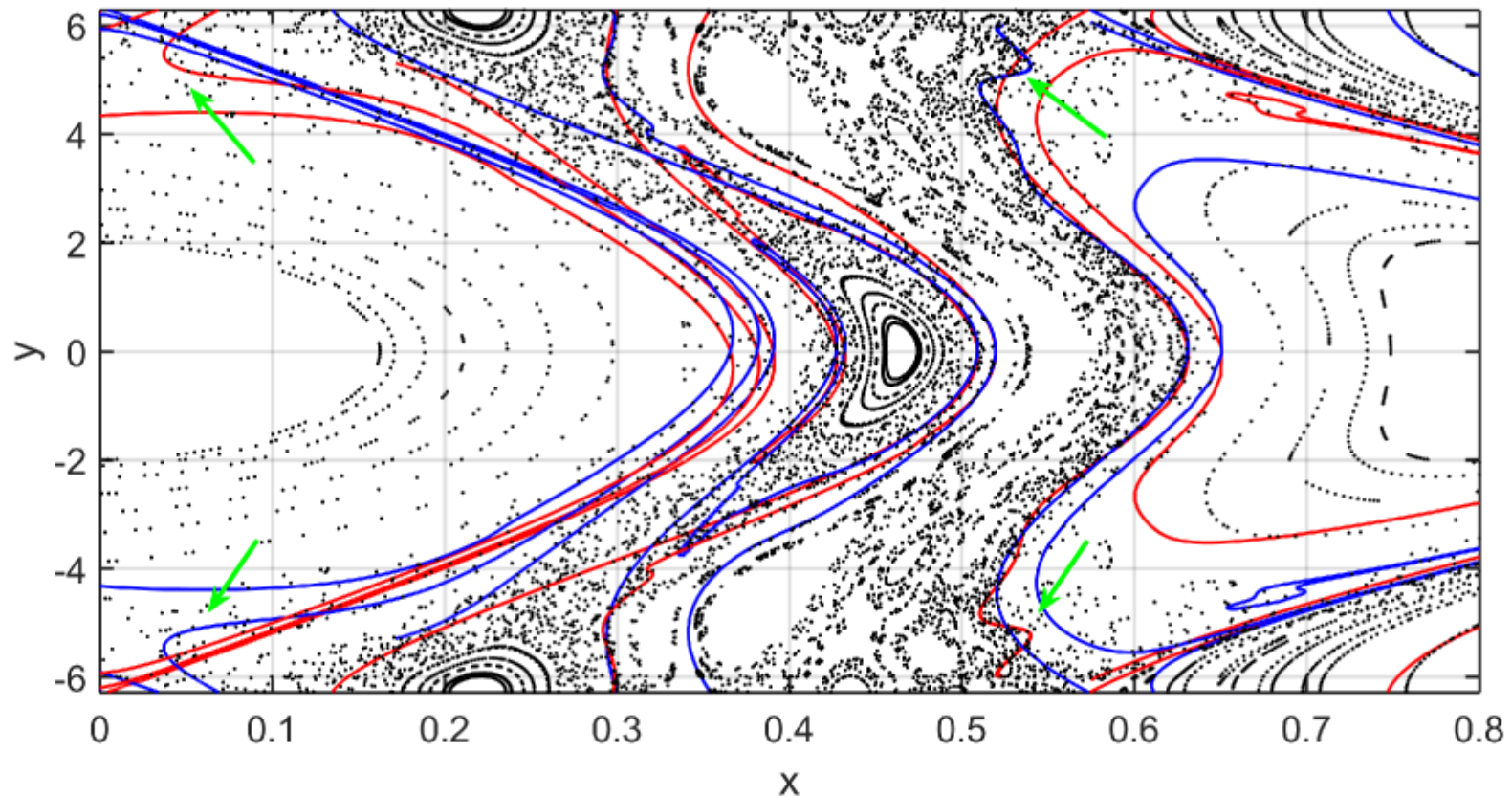
iii) $\xi_{max} \cdot \nabla \lambda_{max} = 0$

the gradient of the largest eigenvalue is along the curve

iv) $\xi_{max} \cdot \nabla^2 \lambda_{max} \cdot \xi_{max} < 0$

repulsive LCS

Most repelling/attracting material lines

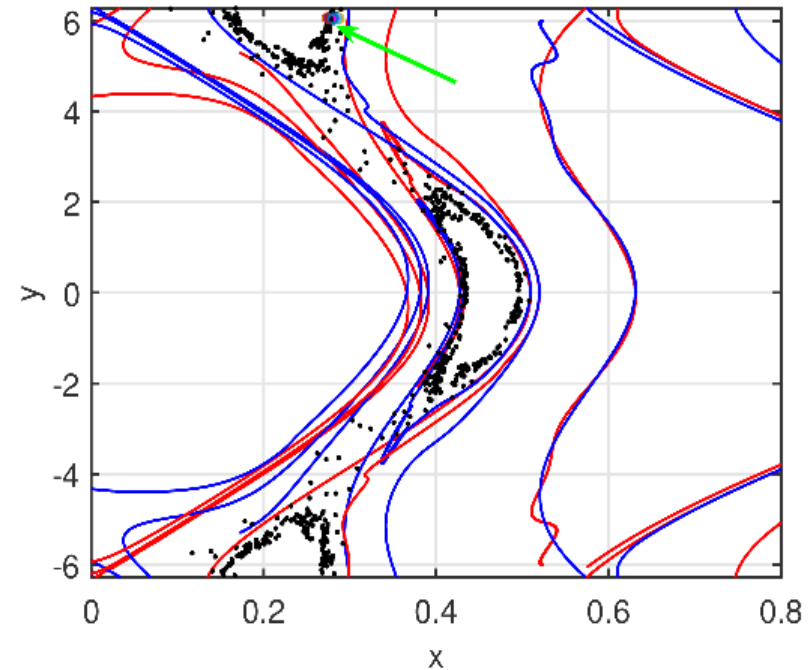
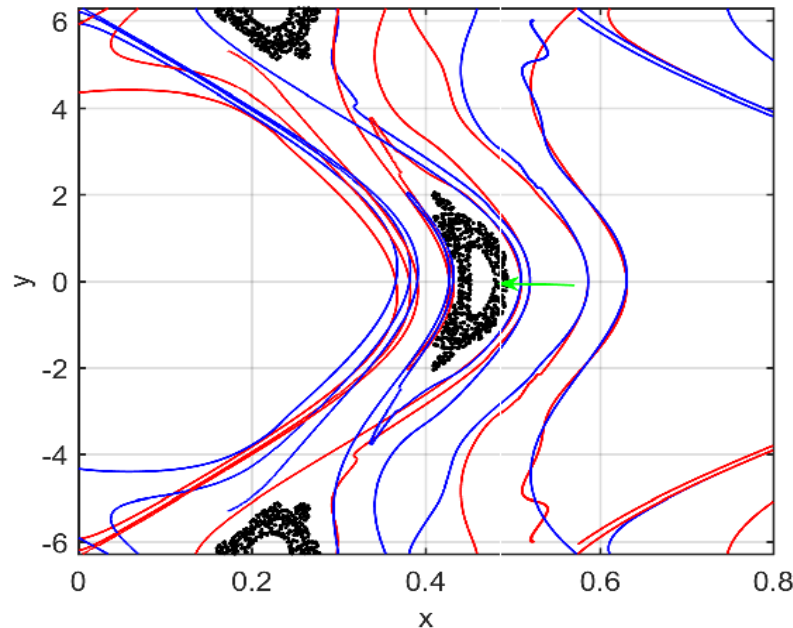


Repelling LCS are in red

Attractive LCS are in blue

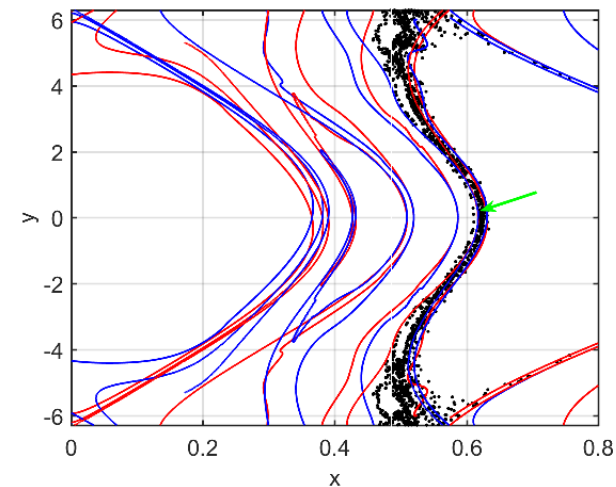
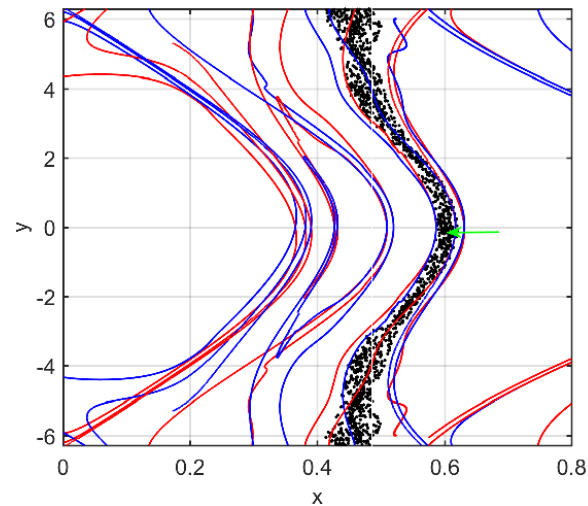
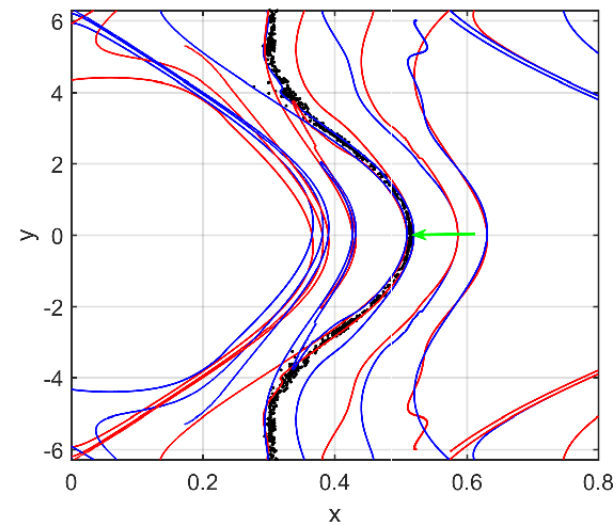
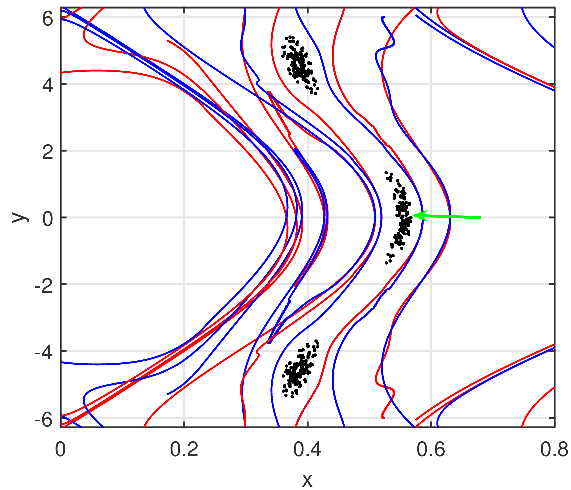
Most repelling/attracting material lines

$80 L_z$

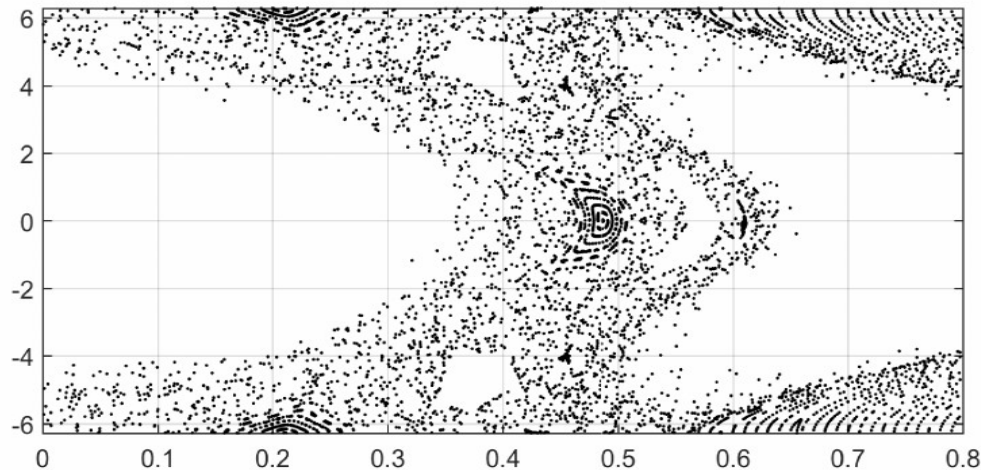


- Since LCS mark the most repelling and attractive material lines they tend to follow respectively the stable and unstable manifolds that in the non autonomous case continue to intersect each other.
- These plot check the robustness of the LCS as barriers to the transport of particles moving along the magnetic field lines.

Most repelling/attracting material lines



The time dependent case



$t = 425$

- What happens when a charged particle moving in the plasma sees a time varying magnetic field during its motion?
- How do the LCS change with the physical time?
- How different are the new LCS from those found at the fixed physical time $t=415$?
- Can the particles cross LCS constructed for particles with different velocities? If yes, how do they cross these LCS?

The time dependent case

- We adopt a simplified model where particles move with a constant velocity along the z-direction only. The new Hamiltonian is given by the modified flux function

$$\psi_v(x, y, z) \equiv \psi(x, y, z, t = (z - z_o)/V)$$

where the superexponential physical time dependence of the flux function between $t=415$ and $t=425$ has been modeled here by interpolating the coefficient of its Fourier expansion according to:

$$\exp \gamma_{k_y, k_z} (t - t_1)^2 \quad \text{for } t > t_1 = 415$$

where we assume the coefficients γ_{k_y, k_z} depend only on the mode numbers k_y and k_z

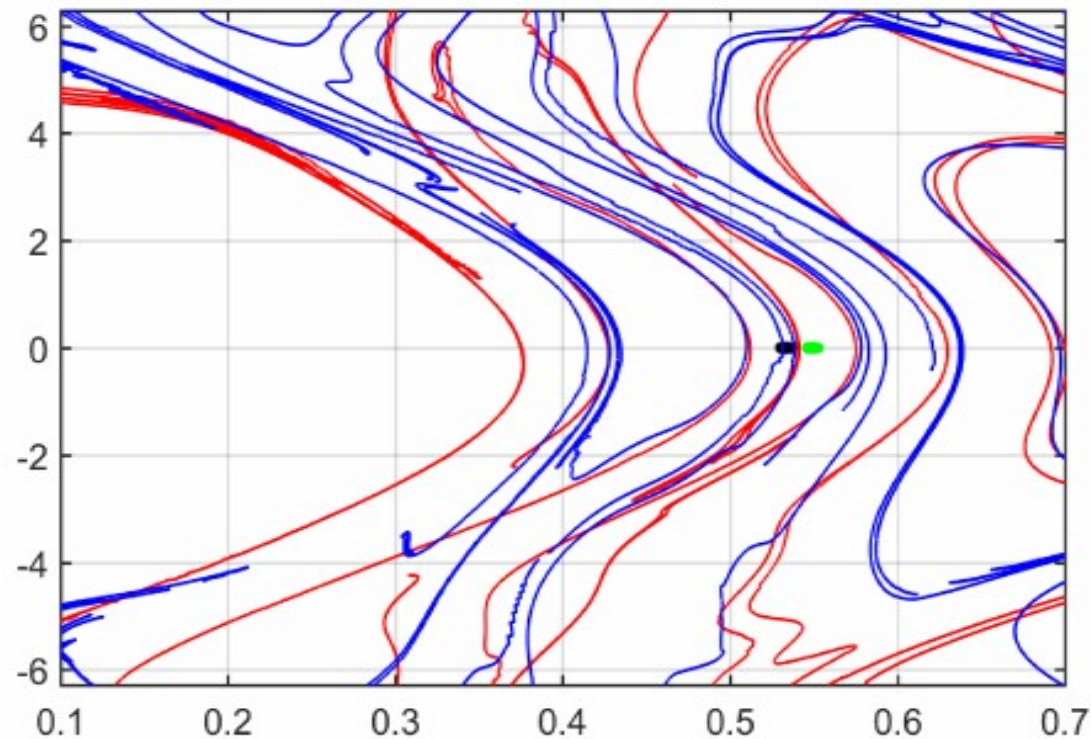
- The dependence of ψ_v on z combines the spatial and time dependence of the magnetic configuration as seen by a particle streaming with velocity V along a field line.

The time dependent case

- We show results obtained for $V=1000$, which corresponds to 10 z-loops of the particles in one time interval. A compromise between having a magnetic field that does not evolve too fast during the motion of particles and being able to show the dependence of the LCS on the velocity and to investigate whether or not the LCS computed for a given velocity act as barriers also for particles with different velocities.
- Note that the fixed time case (or z-periodic case) corresponds to the assumption that the particles move with infinite speed and thus experience a fixed magnetic configuration.

The time dependent case

- Two set of initial conditions are located on the two sides of a repelling LCS.



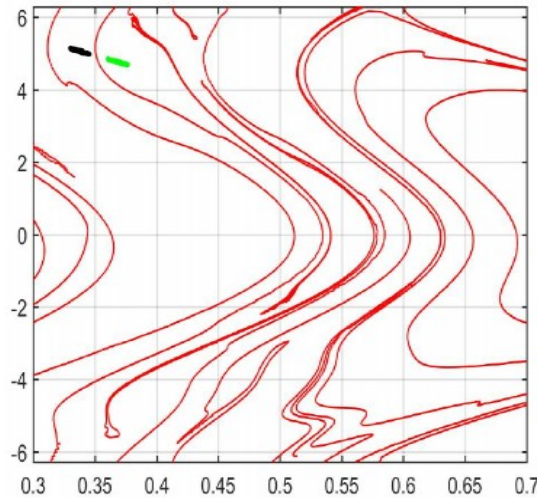
$t = 415$

- How these initial conditions evolve respect to the LCS?

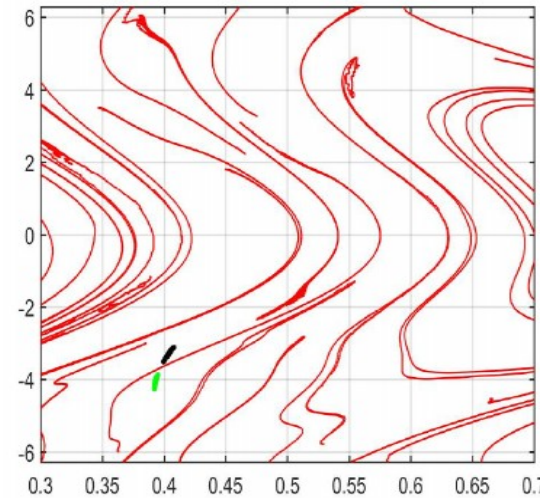
The time dependent case

- Particle move away from the repelling LCS.

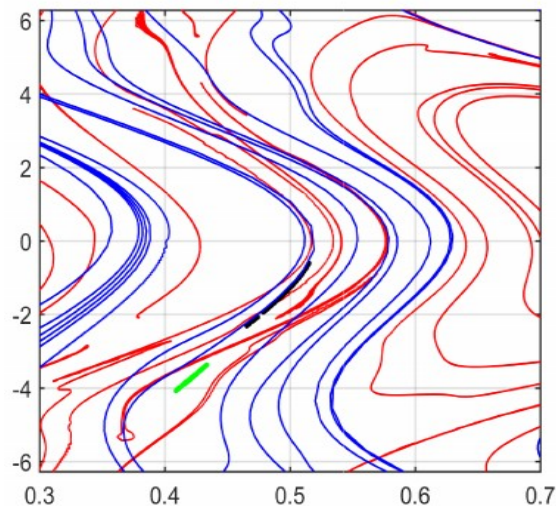
$t = 415.1$



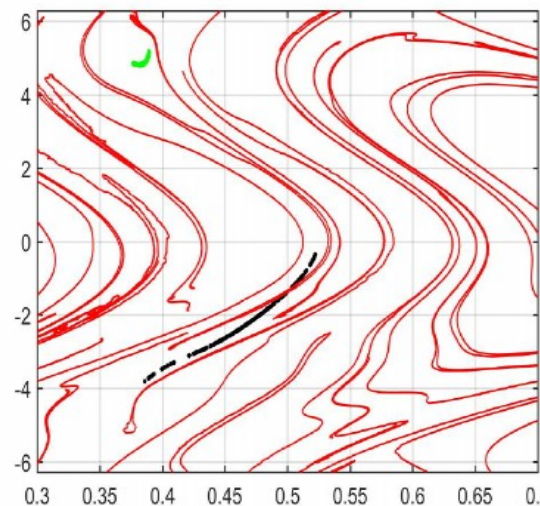
$t = 415.2$



$t = 415.5$



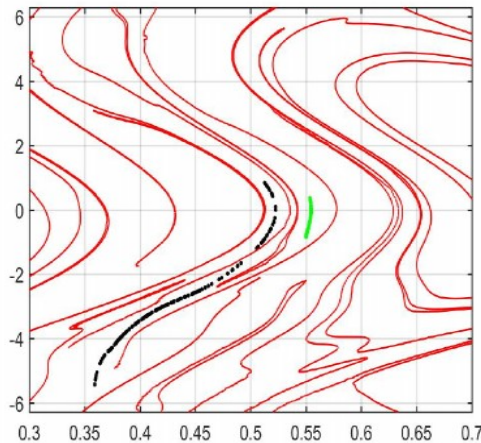
$t = 416$



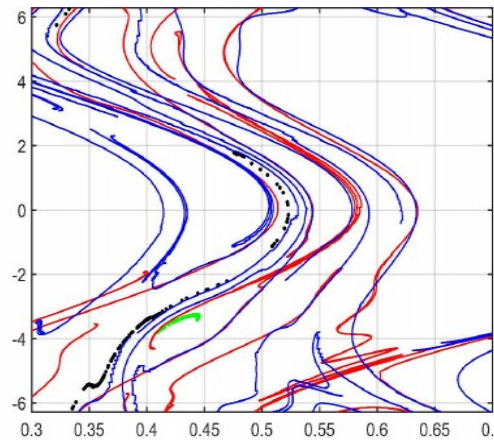
The time dependent case

- Particles close to an attractive LCS are more stretched and convoluted and tend to be aligned with the structures

$t = 416.5$

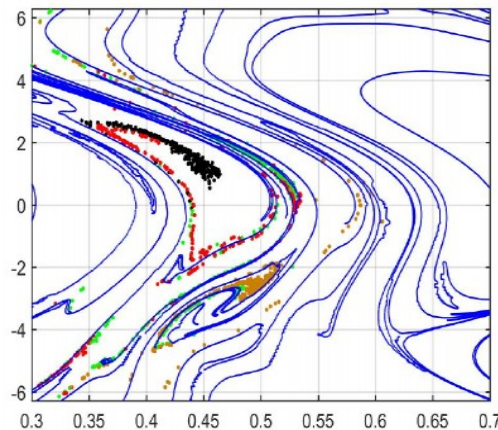


$t = 417$



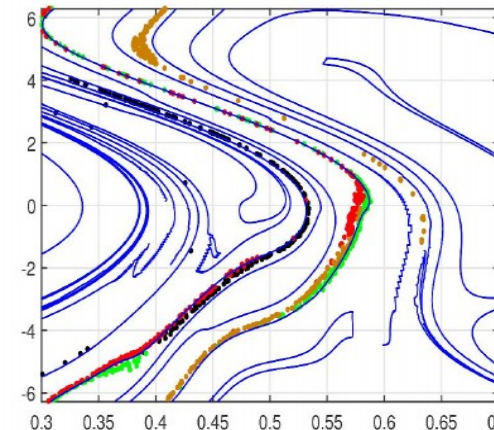
$t = 418$

(particles start
at $t=416$)



$t = 419$

(particles start
at $t=418$)

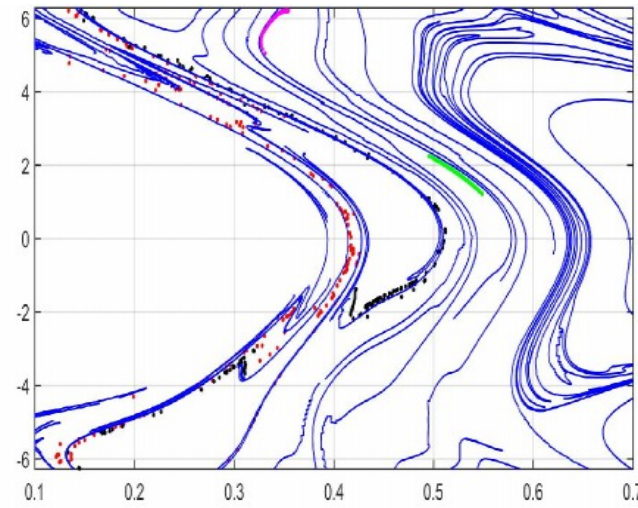
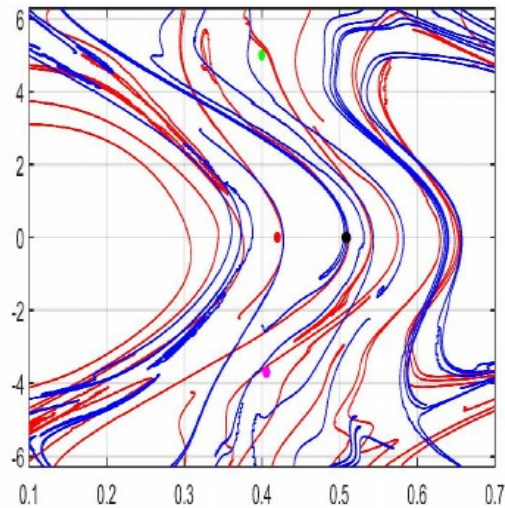


- Although the i.c. cover a wide region, they evolve in such a way so as to position themselves according to attractive LCS. This is the reason why they are referred to as the **skeleton** of chaos.

The time dependent case

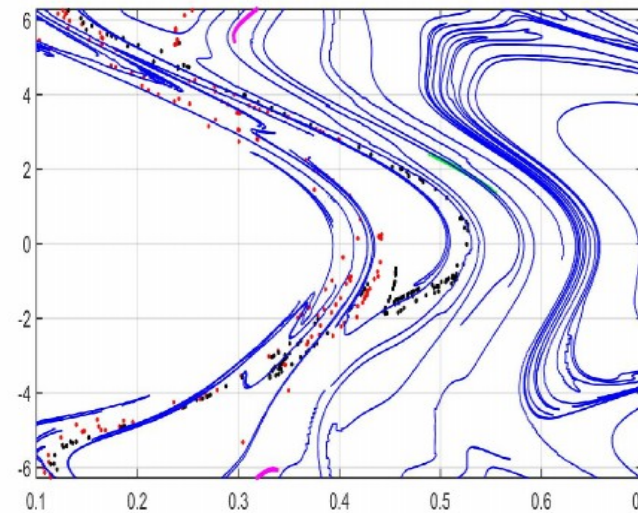
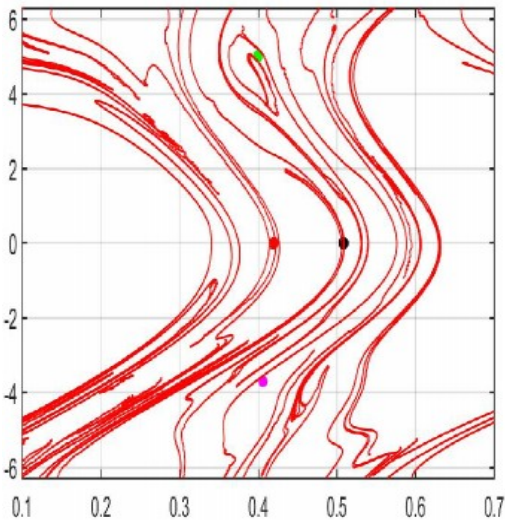
- Particles with different velocity behave differently respect the LCS calculated for $V=1000$.

$V=1000$
 $t=415$



$t=417$

$V=200$
 $t=415$



$t=425$

Conclusions and future perspectives

- Lagrangian Coherent Structures have been shown to provide a very convenient tool in order to identify the main features of the dynamics of charged particles in a magnetized plasma in the presence of a reconnection instability
- Application to the RFX case is on going
- Test electron simulations can test the goodness of the adopted approximations
- Possible application to the study of zonal flows?

Thank you!

Ridges of the FTLE field

Empirical definition: If hiking along a ridge one would expect:

- to be locally at the highest point in the field transverse to the ridge. If the hiker stepped to the left or to the right of the ridge he would step down
- for the topography to drop off steepest in the direction transverse to the ridge. The direction the topography decreases most rapidly should be transverse to the ridge.



Ridges of the FTLE field

- We define first the Finite Time Lyapunov Exponent Field:

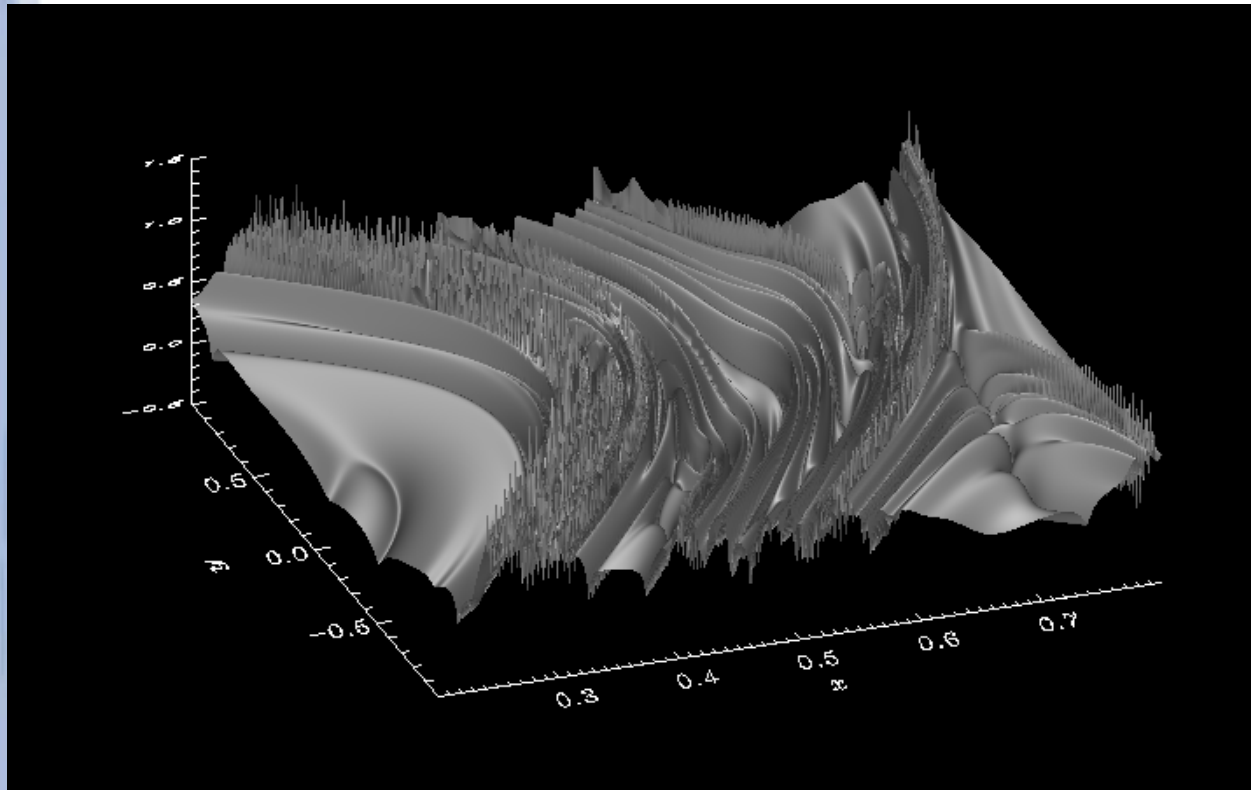
$$\sigma = \sigma_z^Z(x) = \frac{1}{|Z|} \ln \sqrt{\lambda_{\max}}$$

- Z plays the role of an effective finite time interval
- $\lambda_{\max}$ is the largest positive eigenvalue of:

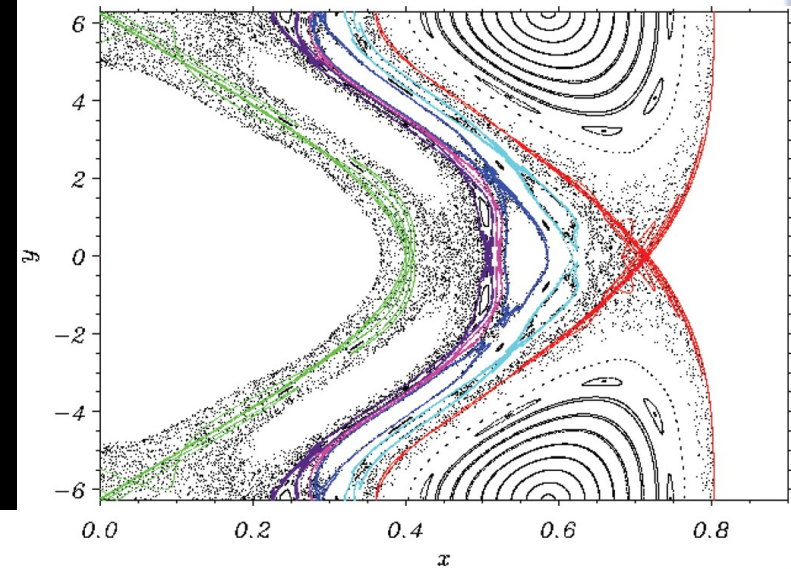
$$M(x, z, Z) = \frac{d\Phi_z^Z(x)}{dx}^\dagger \frac{d\Phi_z^Z(x)}{dx}$$

- $\Phi_z^Z(x)$ is the flow map: $x(z) \rightarrow x(z+Z)$
- In the limit $|Z| \rightarrow \infty$ the standard Lyapunov exponent is recovered.
- We now look for the ridges, defined as curves such that the gradient in the FTLE field is along the curve and such that the second derivative of the largest positive FTLE in the direction perpendicular²⁵ to the curve is minimal.

FTLE field



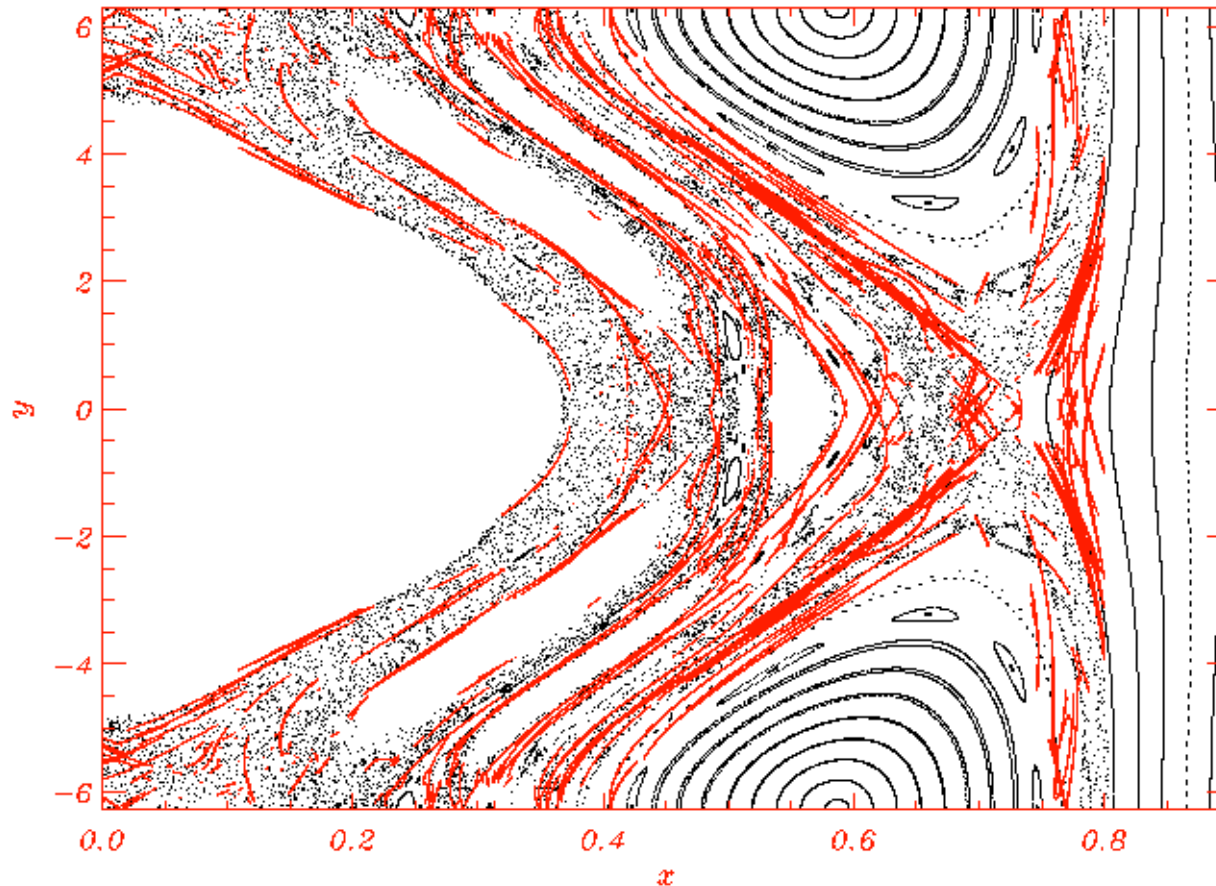
$\sigma(x, y)$ at $Z = 12L_z$



16000X8000 initial conditions
uniformly distributed on the range

$$0 < x < 0.8, -2\pi < y < 2\pi$$

RIDGES

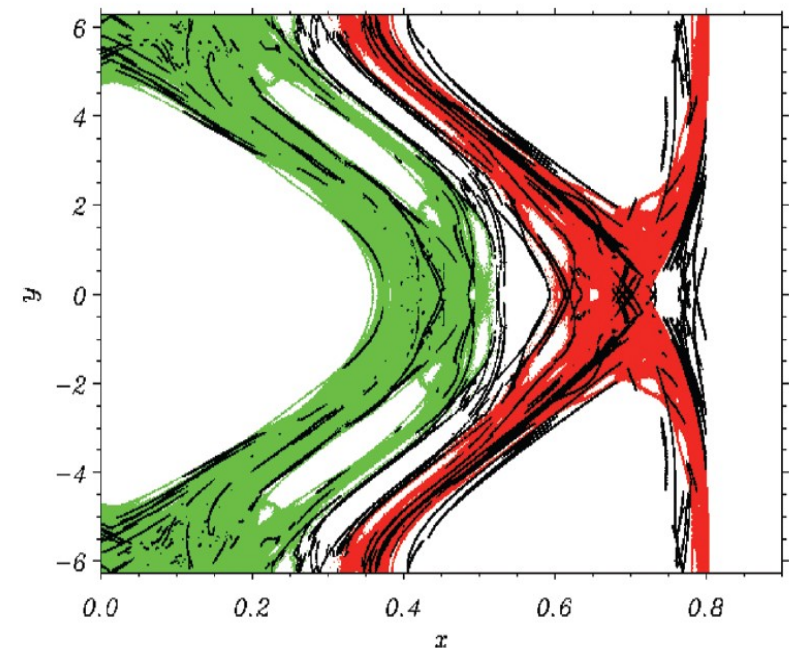
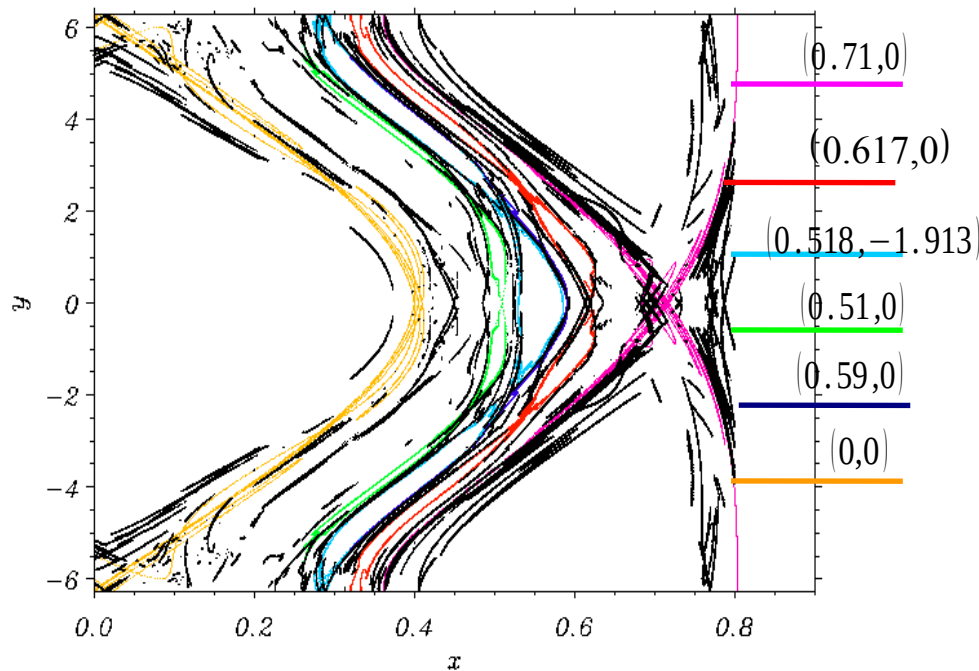


$$Z = 12L_z$$

Borgogno et al, PoP 2011

- Ridges are localised along the boundaries of the chaotic regions
- Well defined structures identified by ridges inside the chaotic domain

Ridges and manifolds



- Ridges (curves in black) lie on the branches of invariant manifolds (curves in color)
- Poincaré plot obtained from 5000 ic iterated for $500L_z$. Both green and red regions are confined by ridges, which confirm the role played by these patterns as magnetic field line transport barriers.

Application to a realistic configuration: results

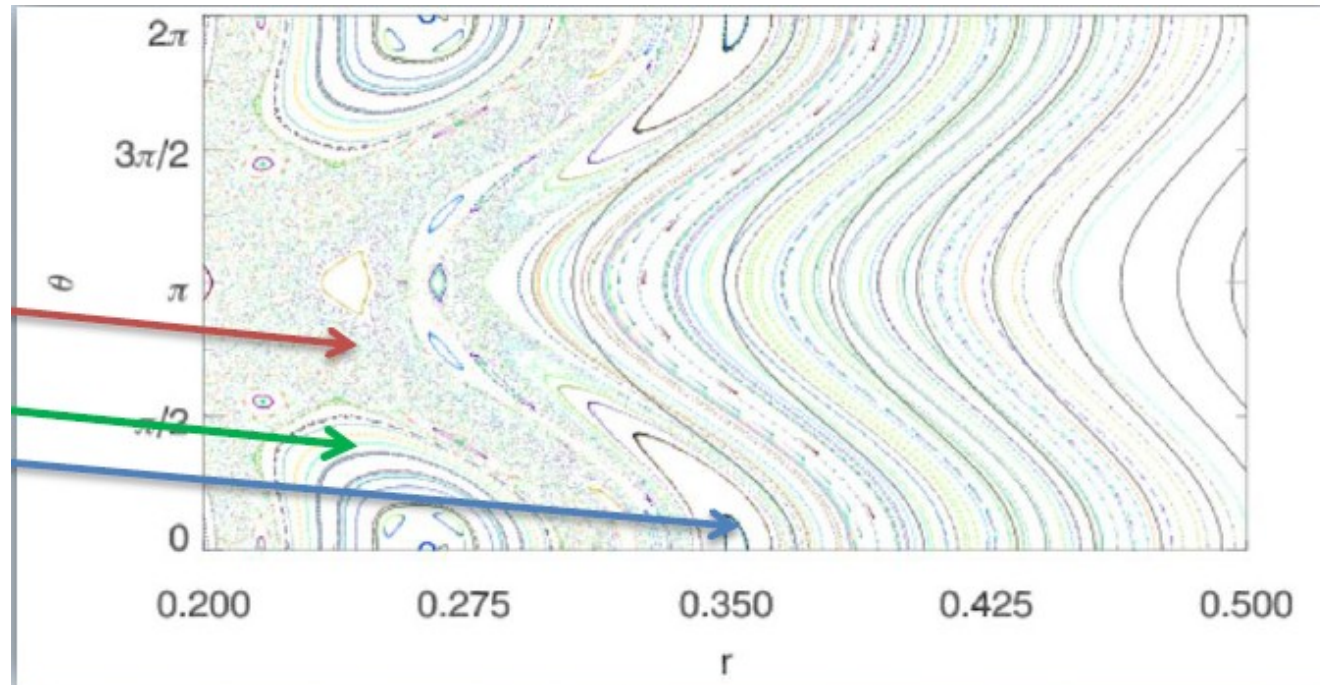
- Numerical simulations of the RFP configuration through SPECYL (3D MHD visco-resistive model) and NEMATO ($d\mathbf{x}/d\tau = \mathbf{B}(\mathbf{x})$) codes.
- **SpeCyl**: 3D MHD visco-resistive numerical simulations
- In the nonlinear phase secondary modes grow and a wide chaotic region develops.
- In the highly nonlinear phase ($t > 2000 \tau_A$) we find a QSH state
- Here we analyze the transition phase ($\approx 600 \tau_A$)

NEMATO: Poincaré map

wide chaotic region

$m=1$ $n=-9$ mode

$m=1$ $n=-10$ mode

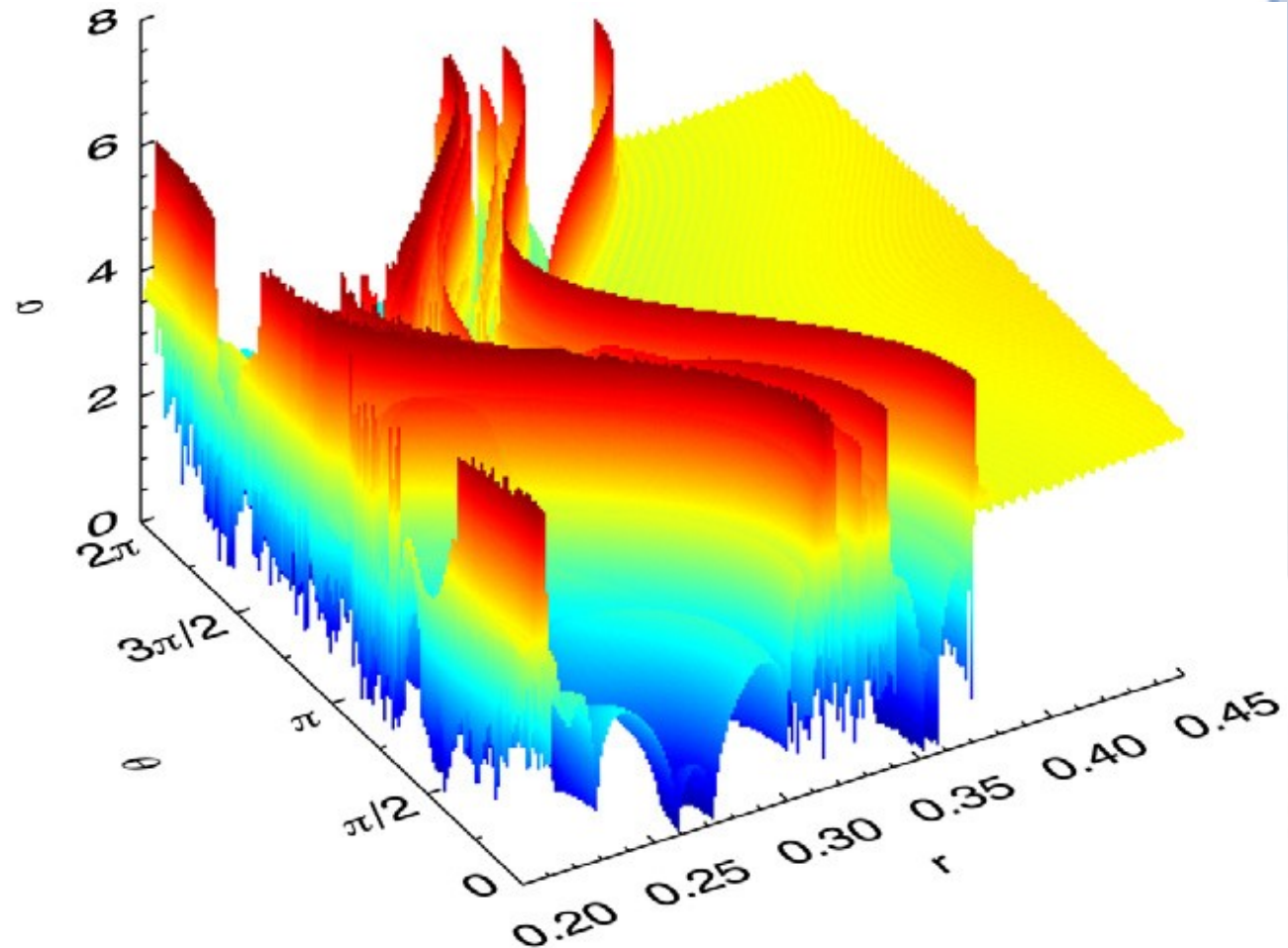


Application to a realistic configuration: results

- Magnetic field line equations:

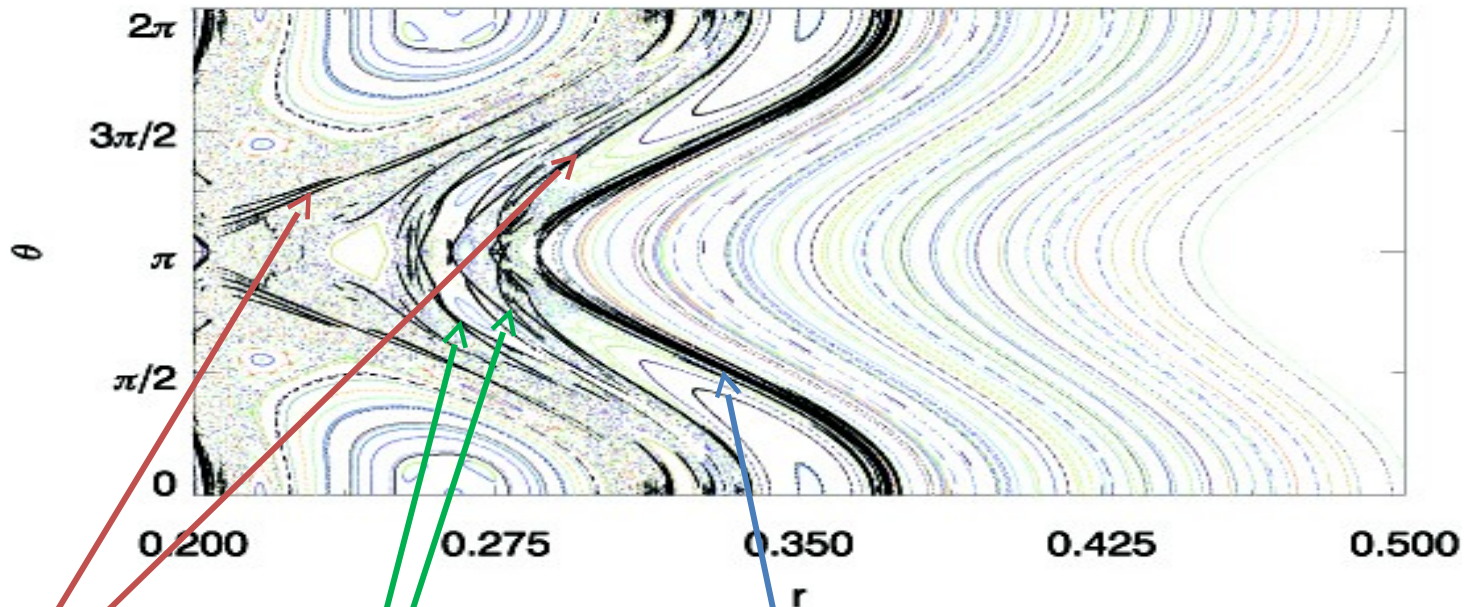
$$\begin{cases} \frac{dr}{dz} = \frac{B_r}{B_z}, \\ \frac{d\theta}{dz} = \frac{1}{r} \frac{B_\theta}{B_z} \end{cases}$$

- FTLE evaluated for $z=10 L_z$, starting from 4096X8400 initial conditions located between $0.2 < r < 0.45$ and $0 < \theta < 2\pi$



- In the chaotic region we observe the presence of ridges
- The FTLE has a regular behavior inside the magnetic islands and where conserved surfaces exist.

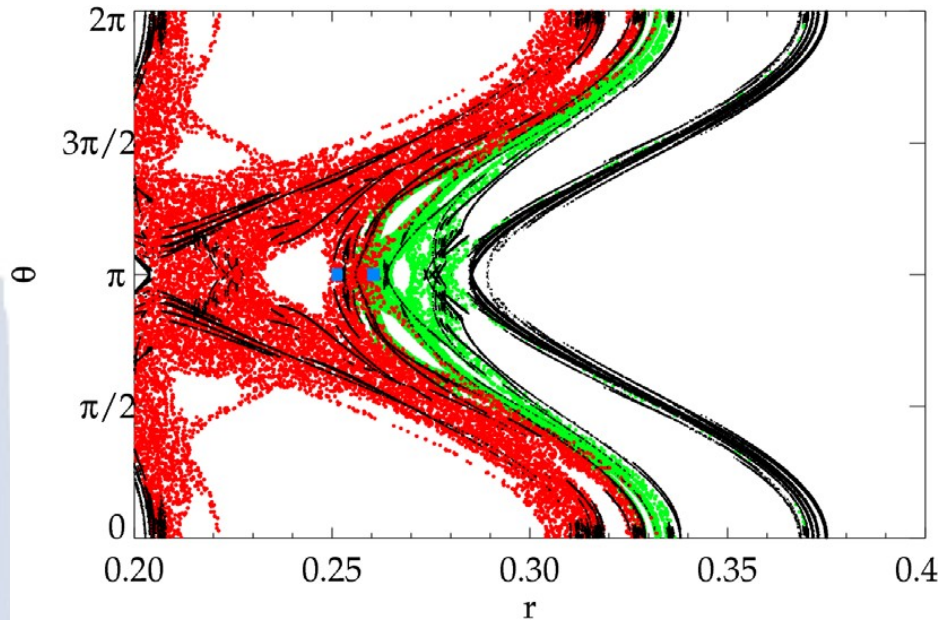
Application to a realistic configuration: results



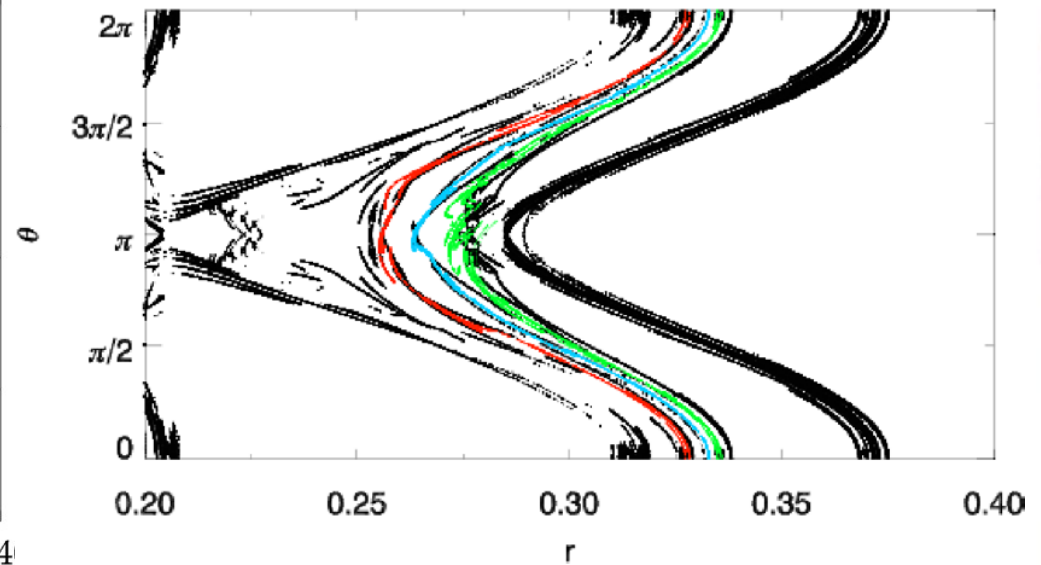
- No LCS are found in regular domains and in highly chaotic regions
- Weak barriers are present at the edge of the magnetic islands
- Strong barrier at the edge of the magnetic flux conserved surfaces

Ridges in the chaotic sea

Application to a realistic configuration: results



Validation by Poincaré plot ($400 L_z$)



Validation by manifolds

Rubino et al, PPCF 2015

Validazione di modelli di tipo Vlasov-(Navier-)Stokes per sistemi bifase

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FISICA DEL PLASMA E MATEMATICA APPLICATA – ROMA, 6/VI/2018

INTRODUZIONE

Costruzione di modelli matematici per miscele contenenti fase dispersa (spray sottili, aerosol) → sistemi bifase
(= fase solida o condensata dispersa in una fase fluida)

sottile = fase dispersa ha frazione volumica/densità di volume trascurabile

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Validazione di sistemi d'equazioni contenenti equazioni cinetiche -tipo Vlasov- e equazioni idrodinamiche -Stokes(S) o Navier-Stokes(NS)- come limiti rigorosi di sistemi microscopici con bassa stocasticità

('microscopici' inteso in senso lato come "scala inferiore")

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Analisi di modelli microscopici originanti, in opportune asintotiche, sistemi descritti da equazioni (macroscopiche) di tipo Vlasov-(Navier-)Stokes

Equazioni limite

$F(t, x, v)$, funzione di distribuzione per la fase dispersa,
 $U(t, x)$ e $\Pi(t, x)$ campi di velocità e pressione del fluido

$$\left\{ \begin{array}{l} \partial_t F + v \cdot \nabla_x F - \kappa \operatorname{div}_v [(v - U)nF] = 0 \\ (\partial_t U + U \cdot \nabla_x U) - \nu \Delta U + \nabla \Pi = (g) + \kappa (j - \rho U), \\ \nabla \cdot U = 0, \end{array} \right. \quad (1)$$

(ν viscosità, κ coefficiente di attrito, n densità del fluido,
 $j = \int v F dv$, $\rho = \int F dv$)

SCHEMI POSSIBILI PER IL SISTEMA A PICCOLA SCALA

"Puro" Microscopico Entrambe le componenti sono descritte da sistemi di particelle interagenti tramite leggi di collisioni deterministiche (es. sfere dure).

Trattazione matematica troppo difficile ($\nexists$ validazione rigorosa per la componente fluida).

Schemi possibili per il sistema a piccola scala

Modello Fluido-Particelle

Fluido: descritto a livello macroscopico (equazioni di S o NS)

Componente dispersa: descritta come sistema di particelle.

Difficoltà dovute alla retroazione particelle-fluido.

Fase dispersa con posizioni stazionarie: L. Desvillettes, F. Golse, V.R. J. Stat. Phys. **131**, 941–967 (2008), V.R. Springer Proceedings Math.& Stat., **162** 285–300 Springer 2016

Schemi possibili per il sistema a piccola scala

Entrambe le componenti sono descritte a livello

"microscopico" (=mesoscopico)

da equazioni di Boltzmann (modello Boltzmann multifase)

E. Bernard, L.Desvillettes, F.Golse, V.R.

Commun.Math.Sci. **15**, 1703–1741 (2017) (Navier-Stokes)

Kinet.Relat.Mod; **11**, 43–69 (2018) (Stokes)

Sistema Fluido-Particelle-FASE DISPERSA

FASE DISPERSA (=PARTICELLE)+ FASE FLUIDA IN DOMINIO REGOLARE
LIMITATO $\Omega \subset \mathbb{R}^3$

Fase dispersa : N sfere rigide identiche di raggio ε , posizione x_k e velocità istantanea v_k .

$B_{x_k, \varepsilon} = \{y \in \mathbb{R}^3 : |y - x_k| < r\}$ volume occupato dalla sfera k .

Misura empirica in x

$$F_N(x, v) = \frac{1}{N} \sum_{k=1}^N \delta_{x_k, v_k}(x, v) \quad (2)$$

Momenti (densità macroscopica e corrente)

$$\rho_N(x) = \int_{\mathbb{R}^3} F_N(x, v) dv; \quad j_N(x) = \int_{\mathbb{R}^3} F_N(x, v) v dv \quad (3)$$

Sistema Fluido-Particelle–fase fluida

Fluido: Descritto dall'equazione di Stokes con aderenza al contorno in $\Omega_\varepsilon = \Omega \setminus \bigcup_{k=1}^N B_{x_k, \varepsilon}$,

$$\left\{ \begin{array}{ll} -\Delta u_\varepsilon + \nabla p_\varepsilon &= g, \quad \text{in } \Omega_\varepsilon \\ \nabla \cdot u_\varepsilon &= 0, \\ u|_{\partial B_{x_k, \varepsilon}} &= v_k, \quad \text{per } k = 1, \dots, N, \\ u|_{\partial \Omega} &= 0. \end{array} \right. \quad (4)$$

$u \equiv u(t, x) \in \mathbb{R}^3$, $p \equiv p(t, x)$ campi di velocità e pressione
 g densità forza esterna per unità di massa, ν viscosità cinematica.

Estensione naturale u_ε a Ω :

$$\bar{u}_\varepsilon(x) = \left\{ \begin{array}{ll} u_\varepsilon(x) & \text{se } x \in \Omega_\varepsilon, \\ v_k & \text{se } x \in B_{x_k, \varepsilon}, \quad k = 1, \dots, N. \end{array} \right. \quad (5)$$

Sistema F-P – APPROSSIMAZIONE DI CAMPO MEDIO

Asintotica: sfere molto piccole, $\varepsilon \ll 1$.

Il numero di sfere deve divergere, $N \rightarrow \infty$ (in modo che quando $\varepsilon \rightarrow 0$ l'effetto collettivo dovuto alle particelle sul fluido non scompaia).

Stima del campo di forza collettivo (ν viscosità cinematica)

Resistenza di una sfera in un fluido di Stokes di densità ρ_f :

$$6\pi\rho_f\nu\varepsilon V$$

V velocità relativa sfera risp. velocità del fluido a ∞

Campo di forze collettivo per un sistema di N sfere identiche:

$$6\pi\rho_f\nu N\varepsilon\langle V \rangle,$$

$(\langle V \rangle = \frac{1}{N} \sum_{i=1}^N V_i$ velocità media delle sfere)

Limite di campo medio

Si vorrebbe

$$6\pi\rho_f\nu N\varepsilon\langle V\rangle \sim F$$

assumendo che $\rho_f, \nu, \langle V\rangle$ siano $O(1)$.

Si richiede quindi, per $N \gg 1, \varepsilon \ll 1$,

$$N\varepsilon = \text{const}$$

- La distanza tra le particelle si assume essere sufficientemente grande da permettere che la formula di Stokes sia valida per ogni singola particella (le particelle non interagiscono attraverso il flusso e la forza d'attrito collettiva è la somma delle forze di attrito dovute alle singole particelle).

$$\inf_{1 \leq k \neq l \leq N} |x_k - x_l| > 2r_\varepsilon \text{ con } r_\varepsilon := \varepsilon^{1/3}. \quad (6)$$

(consistente con la scala critica data in P.-E. Jabin, F. Otto. Commun. Math. Phys. **250** (2004), 415–432.)

- Le particelle non interagiscono direttamente con la frontiera di Ω

$$\inf_{1 \leq k \leq N} \text{dist}(x_k, \partial\Omega) > r_\varepsilon. \quad (7)$$

Enunciato del teorema

Teorema 1 *Si consideri un sistema di N sfere $B_{x_k, \varepsilon}$, $k = 1, \dots, N$, e $\varepsilon = 1/N$ contenute in $\Omega \subset \mathbb{R}^3$ regolare e limitato t.c.*

- 1. Valgono le ipotesi di non sovrapposizione (6)-(7)*
- 2. $\sup_{N \geq 1} \iint_{\Omega \times \mathbb{R}^3} \frac{1}{2} |v|^2 F_N(x, v) dx dv < \infty$*
- 3. densità macroscopica e corrente convergono debolmente nel senso delle misure*

*$\rho_N \rightharpoonup \rho \in C(\bar{\Omega})$, $j_N \rightharpoonup j \in C(\bar{\Omega})$ per $N \rightarrow \infty$
 $\forall g \in (L^2(\Omega))^3$, sia u_ε l'unica soluzione in $(H^1(\Omega_\varepsilon))^3$
dell'equazione di Stokes (4), e $\bar{u}_\varepsilon$ la sua estensione naturale
data in (5). Allora, $\bar{u}_\varepsilon \xrightarrow{(L^2(\Omega))^3} U$ soluzione di*

$$\left\{ \begin{array}{lcl} -\Delta U + \nabla \Pi & = & g + 6\pi(j - \rho U), \\ \nabla \cdot U & = & 0, \\ U|_{\partial\Omega} & = & 0 \end{array} \right. \quad (8)$$

Cenni sulla dimostrazione

Dimostrazione: metodi di omogeneizzazione

La dimostrazione utilizza il metodo utilizzato, per distribuzioni periodiche di ostacoli con condizioni di Dirichlet, in

1. D. Cioranescu, F. Murat Collège de France Seminar, Vol. 2, Research Notes in Mathematics, **60**, p.98-138, (1982)
(Laplace equation)
2. G. Allaire. Arch. Rational Mech. Anal., **113**, p. 209-259, 1991.
(Stokes, Navier-Stokes equations)

Teorema analogo per sistema limite con equazione di Navier-Stokes.

Differentemente da [2]: correttori solenoidali $\Rightarrow$ stime della pressione non necessarie

Cenni sulla dimostrazione

Il comportamento asintotico (omogeneizzazione) è regolato dalla **capacità** associata allo spazio occupato dalle inclusioni, quantità che "misura" la grandezza effettiva dell'insieme occupato dalle inclusioni.

$$\Sigma \subset S \subset \mathbb{R}^d$$

$$\text{Cap}_S(\Sigma) = \inf \{ \|\nabla f\|_{L^2(S)}^2 : H_0^1(S) \ni f \geq 1 \text{ q.o. in un intorno di } \Sigma \},$$

(es. V.R. Proceedings PSPDEIII, 285-300 (2016))

Particelle ben separate (prima ipotesi tecnica, (6)) $\implies$ le particelle non possono muoversi.

Sistema limite per modello Boltzmann multifase

$F(t, x, v)$ distribuzione di particelle (fase dispersa)

$u(t, x)$ and $p(t, x)$ campi di pressione e velocità del fluido

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \frac{\kappa}{m_p} \operatorname{div}_v((v - u)F) = 0, \\ [\rho_g(\partial_t u + u \cdot \nabla_x u)] + \nabla_x p = \rho_g \nu \Delta_x u + \kappa \int_{\mathbf{R}^3} (v - u)F \, dv, \\ \operatorname{div}_x u = 0. \end{cases} \quad (9)$$

[] NS $\not\ll \not\gg$ S

PARAMETRI

ν viscosità κ coefficiente d'attrito

ρ_g densità del fluido m_p massa delle particelle della fase dispersa

MODELLO BOLTZMANN MULTIFASE

$F(t, x, v)$ distribuzione di particelle (componente dispersa)

$f(t, x, w)$ distribuzione delle molecole del gas (fluido)

$$(\partial_t F + v \cdot \nabla_x F) = \mathcal{D}(F, f) + \mathcal{B}(F)$$

$$(\partial_t f + w \cdot \nabla_x f) = \mathcal{R}(f, F) + \mathcal{C}(f)$$

- $\mathcal{D}(F, f)$ **deflessione** delle particelle da parte delle molecole
- $\mathcal{R}(f, F)$ **attrito** esercitato sulle molecole di gas dalle particelle
- $\mathcal{C}(f), \mathcal{B}(F)$ **integrale di collisione** risp. per coppie di molecole di gas e coppie di particelle.

- Conservazione locale di massa, momento, energia (coll. tra molecole)

$$\int_{\mathbf{R}^3} \mathcal{C}(f)(w) \begin{pmatrix} 1 \\ w \\ |w|^2 \end{pmatrix} dw = 0. \quad (10)$$

- Conservazione locale di massa, momento, energia (coll. tra molecole)

$$\int_{\mathbf{R}^3} \mathcal{C}(f)(w) \begin{pmatrix} 1 \\ w \\ |w|^2 \end{pmatrix} dw = 0. \quad (10)$$

- Conservazione locale di massa e momento (coll. tra particelle)

$$\int_{\mathbf{R}^3} \mathcal{B}(F)(v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv = 0. \quad (11)$$

- Conservazione locale di massa, momento, energia (coll. tra molecole)

$$\int_{\mathbf{R}^3} \mathcal{C}(f)(w) \begin{pmatrix} 1 \\ w \\ |w|^2 \end{pmatrix} dw = 0. \quad (10)$$

- Conservazione locale di massa e momento (coll. tra particelle)

$$\int_{\mathbf{R}^3} \mathcal{B}(F)(v) \begin{pmatrix} 1 \\ v \end{pmatrix} dv = 0. \quad (11)$$

- Conservazione locale del numero di particelle per specie

$$\int_{\mathbf{R}^3} \mathcal{D}(F, f)(v) dv = \int_{\mathbf{R}^3} \mathcal{R}(f, F)(w) dw = 0. \quad (12)$$

■ Conservazione locale del momento

$$m_p \int_{\mathbf{R}^3} \mathcal{D}(F, f)(v) v \, dv + m_g \int_{\mathbf{R}^3} \mathcal{R}(f, F)(w) w \, dw = 0, \quad (13)$$

■ Conservazione locale del momento

$$m_p \int_{\mathbf{R}^3} \mathcal{D}(F, f)(v) v \, dv + m_g \int_{\mathbf{R}^3} \mathcal{R}(f, F)(w) w \, dw = 0, \quad (13)$$

■ Conservazione locale dell'energia (se $\mathcal{D}, \mathcal{R}$ elastiche)

$$m_p \int_{\mathbf{R}^3} \mathcal{D}(F, f)(v) \frac{1}{2} |v|^2 \, dv + m_g \int_{\mathbf{R}^3} \mathcal{R}(f, F)(w) \frac{1}{2} |w|^2 \, dw = 0.$$

$$\begin{aligned}
 \mathcal{B}(F)(v) &= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} F(v') F(v'_*) \Pi_{pp}(v, dv' dv'_*) \\
 &\quad - F(v) \int_{\mathbf{R}^3} F(v_*) |v - v_*| \Sigma_{pp}(|v - v_*|) dv_* , \\
 \mathcal{C}(f)(w) &= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} f(w') f(w'_*) \Pi_{gg}(w, dw' dw'_*) \\
 &\quad - f(w) \int_{\mathbf{R}^3} f(w_*) |w - w_*| \Sigma_{gg}(|w - w_*|) dw_* ,
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 \mathcal{D}(F, f)(v) &= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} F(v') f(w') \Pi_{pg}(v, dv' dw') \\
 &\quad - F(v) \int_{\mathbf{R}^3} f(w) |v - w| \Sigma_{pg}(|v - w|) dw, \\
 \mathcal{R}(f, F)(w) &= \iint_{\mathbf{R}^3 \times \mathbf{R}^3} F(v') f(w') \Pi_{gp}(w, dv' dw') \\
 &\quad - f(w) \int_{\mathbf{R}^3} F(v) |v - w| \Sigma_{pg}(|v - w|) dv.
 \end{aligned} \tag{15}$$

Π_* vs. Σ_*

$$\begin{aligned}\int_{\mathbf{R}_v^3} dv \Pi_{pp}(v, dv' dv'_*) &= |v' - v'_*| \Sigma_{pp}(|v' - v'_*|) dv' dv'_*, \\ \int_{\mathbf{R}_w^3} dw \Pi_{gg}(w, dw' dw'_*) &= |w' - w'_*| \Sigma_{gg}(|w' - w'_*|) dw' dw'_*, \\ \int_{\mathbf{R}_v^3} dv \Pi_{pg}(v, dv' dw') &= |v' - w'| \Sigma_{pg}(|v' - w'|) dv' dw', \\ \int_{\mathbf{R}_w^3} dw \Pi_{gp}(w, dv' dw') &= |v' - w'| \Sigma_{pg}(|v' - w'|) dv' dw'.\end{aligned}\tag{16}$$

MODELLO BOLTZMANN MULTIFASE - FORMA ADIMENSIONALE

Parametro	Definizione
L	taglia del recipiente (scatola periodica)
$\mathcal{N}_p$	numero di particelle / L^3
$\mathcal{N}_g$	numero di molecole di gas / L^3
V_p	velocità termica delle particelle
V_g	velocità termica delle molecole di gas
S_{pp}	sez. d'urto media particella/particella
S_{pg}	sez. d'urto media particella/gas
S_{gg}	sez. d'urto media molecolare
$\eta = m_g/m_p$	rapporto di massa (molecole/particelle)
$\mu = (m_g \mathcal{N}_g)/(m_p \mathcal{N}_p)$	frazione di massa (gas/particelle)
$\epsilon = V_p/V_g$	rapp. di veloc. termica (particelle/molecole)

Variabili dipendenti e indipendenti riscalate:

- Spazio: $\hat{x} = \frac{x}{L}$

- Tempo: $\hat{t} = \frac{V_p}{L} t$

- Velocità particelle: $\hat{v} = \frac{v}{V_p}$

- Velocità molecole: $\hat{w} = \frac{w}{V_g}$

- Densità particelle: $\hat{F} = \frac{V_p^3}{N_p} F$

- Densità molecole: $\hat{f} = \frac{V_g^3}{N_g} f$

NUCLEI DI COLLISIONE RISCALATI

$$\begin{aligned}\hat{\Pi}_{pp}(\hat{v}, d\hat{v}' d\hat{v}'_*) &= \Pi_{pp}(v, dv' dv'_*)/S_{pp}V_p^4, \\ \hat{\Pi}_{gg}(\hat{w}, d\hat{w}' d\hat{w}'_*) &= \Pi_{gg}(w, dw' dw'_*)/S_{gg}V_g^4, \\ \hat{\Pi}_{pg}(\hat{v}, d\hat{v}' d\hat{w}') &= \Pi_{pg}(v, dv' dw')/S_{pg}V_g^4, \\ \hat{\Pi}_{gp}(\hat{w}, d\hat{v}' d\hat{w}') &= \Pi_{gp}(w, dv' dw')/S_{pg}V_gV_p^3.\end{aligned}$$

$$\begin{aligned}\hat{\Sigma}_{pp}(|\hat{v}|) &= \Sigma_{pp}(V_p|\hat{v}|)/S_{pp}, \\ \hat{\Sigma}_{gg}(|\hat{w}|) &= \Sigma_{gg}(V_g|\hat{w}|)/S_{gg}, \\ \hat{\Sigma}_{pg}(|\hat{z}|) &= \Sigma_{pg}(V_g|\hat{z}|)/S_{pg}.\end{aligned}$$

MODELLO BOLTZMANN MULTIFASE

Sistema di equazioni Boltzmann multifase riscaldati

$$(\partial_{\hat{t}}(\hat{F} + \hat{v} \cdot \nabla_{\hat{x}} \hat{F})) = \mathcal{N}_g S_{pg} L \frac{V_g}{V_p} \mathcal{D}(\hat{F}, \hat{f}) + \mathcal{N}_p S_{pp} L \hat{\mathcal{B}}(\hat{F})$$

$$(\partial_{\hat{t}} \hat{f} + \hat{w} \cdot \nabla_{\hat{x}} \hat{f}) = N_p S_{pg} L \frac{V_g}{V_p} \mathcal{R}(\hat{f}, \hat{F}) + N_g S_{gg} L \frac{V_g}{V_p} \mathcal{C}(\hat{f})$$

con

$(N_g S_{gg})^{-1}$ cammino libero medio per le collisioni tra molecole

$((N_p S_{pp})^{-1})$ cammino libero medio per le collisioni tra particelle
se fossero prese in considerazione)

Si assume $\mathcal{N}_p S_{pp} L \ll 1$

MODELLO BOLTZMANN MULTIFASE-SCALING

Possibili scaling:

Vlasov-Navier-Stokes

$$\frac{V_p}{V_g} = N_p S_{pg} L = (N_g S_{gg} L)^{-1} = \epsilon \ll 1 \quad \frac{N_p}{N_g} = \eta \ll \epsilon^2$$

(densità di massa gas/densità di massa particelle ≈ 1)

Vlasov-Stokes

$$\frac{V_p}{V_g} = \epsilon \quad \frac{N_p}{N_g} = \eta/\mu \quad N_g S_{gg} L = \frac{1}{\lambda} \quad N_p S_{pg} L = \epsilon/\mu \quad \epsilon \ll \lambda \ll 1 \quad \mu = \frac{\epsilon}{\lambda}$$

(densità di massa gas/densità di massa particelle $\ll 1$)

MODELLO BOLTZMANN MULTIFASE

Riguardo le collisioni particella-particella:

$\mathcal{B}(F)$ integrale di collisione tra particelle Boltzmann

$$\mathcal{B} \implies N_p S_{pp} V_p \mathcal{B}$$

Il termine corrispondente nella prima delle equazioni riscalate è

$$N_p S_{pp} L \mathcal{B}$$

Riguardo le collisioni particella-particella:

Se $S_{pp} = O(S_{pg})$ si ha

$$N_p S_{pp} L = O(N_p S_{pg} L)$$

quindi

- scaling Navier-Stokes: $N_p S_{pp} L = O(\epsilon)$
- scaling Stokes: $N_p S_{pp} L = O(\frac{\epsilon}{\mu}) = O(\lambda)$

Le equazioni riscalate nei due scaling sono quindi
Vlasov-Navier-Stokes

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\eta} \mathcal{D}(F, f)$$

$$\partial_t f + \frac{1}{\epsilon} w \cdot \nabla_x f = \mathcal{R}(f, F) + \frac{1}{\epsilon^2} \mathcal{C}(f)$$

Vlasov–Stokes

$$\partial_t F + v \cdot \nabla_x F = \frac{1}{\eta} \mathcal{D}(F, f)$$

$$\partial_t f + \frac{1}{\epsilon} w \cdot \nabla_x f = \frac{1}{\mu} \mathcal{R}(f, F) + \frac{1}{\lambda \epsilon} \mathcal{C}(f)$$

$$\begin{aligned} \mathcal{C}(f)(w) = & \iint_{\mathbf{R}^3 \times \mathbf{R}^3} f(w') f(w'_*) \Pi_{gg}(w, dw' dw'_*) \\ & - f(w) \int_{\mathbf{R}^3} f(w_*) |w - w_*| \Sigma_{gg}(|w - w_*|) dw_* , \end{aligned} \quad (17)$$

$$\begin{aligned} \mathcal{D}(F, f)(v) = & \iint_{\mathbf{R}^3 \times \mathbf{R}^3} F(V) f(W) \Pi_{pg}(v, dV dW) \\ & - F(v) \int_{\mathbf{R}^3} f(w) |\epsilon v - w| \Sigma_{pg}(|\epsilon v - w|) dw , \end{aligned} \quad (18)$$

$$\begin{aligned} \mathcal{R}(f, F)(w) = & \iint_{\mathbf{R}^3 \times \mathbf{R}^3} F(V) f(W) \Pi_{gp}(w, dV dW) \\ & - f(w) \int_{\mathbf{R}^3} F(v) |\epsilon v - w| \Sigma_{gp}(|\epsilon v - w|) dv , \end{aligned} \quad (19)$$

Integrale di Collisione molecola-molecola

$$\mathcal{C}(f)(w) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (f(w')f(w'_*) - f(w)f(w_*))c(w - w_*, \omega) dw_* d\omega, \quad (20)$$

$$\begin{aligned} w' &\equiv w'(w, w_*, \omega) := w - (w - w_*) \cdot \omega \omega, \\ w'_* &\equiv w'_*(w, w_*, \omega) := w_* + (w - w_*) \cdot \omega \omega, \end{aligned} \quad (21)$$

(c corrispondente a potenziali duri, con cut off di Grad, cfr. Ip. (H6))

IPOTESI SU Π_{pg}, Π_{gp}

■ **Ipotesi (H1).** $[\Sigma_{pg} = \Sigma_{gp}]$

$\exists q$ misurabile $0 \leq q(r) \leq C(1+r)$, $C > 0$ t. c.

$$\int_{\mathbf{R}^3} \Pi_{pg}(v, dV dW) dv = \int_{\mathbf{R}^3} \Pi_{gp}(w, dV dW) dw = q(|\epsilon V - W|) dV dW .$$

IPOTESI SU Π_{pg}, Π_{gp}

■ **Ipotesi (H1).** $[\Sigma_{pg} = \Sigma_{gp}]$

$\exists q$ misurabile $0 \leq q(r) \leq C(1+r)$, $C > 0$ t. c.

$$\int_{\mathbf{R}^3} \Pi_{pg}(v, dV dW) dv = \int_{\mathbf{R}^3} \Pi_{gp}(w, dV dW) dw = q(|\epsilon V - W|) dV dW.$$

■ **Ipotesi (H2).** CONSERVAZIONE DEL MOMENTO COLLISIONI M-P

$\exists Q \equiv Q(r) \in C(\mathbf{R}_+^*)$, con $Q \geq 0$ e $Q(r) + |Q'(r)| \leq C(1+r)$
per $C > 0$, t.c.

$$\begin{aligned} \epsilon \int_{\mathbf{R}^3} dv (v - V) \Pi_{pg}(v, dV dW) &= -\eta \int_{\mathbf{R}^3} dw (w - W) \Pi_{gp}(w, dV dW) \\ &= -\frac{\eta}{1+\eta} (\epsilon V - W) Q(|\epsilon V - W|) dV dW. \end{aligned}$$

IPOTESI SU Π_{pg}, Π_{gp}

Ipotesi (H3). $\exists C > 0$ t. c.

$$\int_{\mathbf{R}^3} dv \left| \epsilon v - \frac{\epsilon V + \eta W}{1 + \eta} \right|^2 \Pi_{pg}(v, dV dW) \leq \\ C \eta^2 (1 + |\epsilon V - W|^2) q(|\epsilon V - W|) dV dW,$$

IPOTESI SU Π_{pg}, Π_{gp} -NAVIER-STOKES

Ipotesi (H4). La misura limite $\Pi_{gp}^{0,0}$ è t.c.:

$$\mathcal{T}_R \# \Pi_{gp}^{0,0} = \Pi_{gp}^{0,0} \quad \forall R \in O_3(\mathbf{R}), \text{ con}$$

$$\mathcal{T}_R : (w, V, W) \mapsto (Rw, V, RW),$$

$$\text{e, } \forall \Phi(w, W) \text{ t. c. } |\Phi(w, W)| \leq C(1 + |w|^2 + |W|^2)M(W),$$

$$\exists p > 3 \text{ t.c.}$$

$$\int_{\mathbf{R}^3} (1 + |V|^2)^{-p} \left| \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Phi(w, W) (\Pi_{gp}^{\epsilon, \eta}(w, dV dW) - \Pi_{gp}^{0,0}(w, dV dW)) dw \right|$$

$$\rightarrow 0$$

quando $\epsilon, \eta \rightarrow 0$. Inoltre,

$$\iint_{\mathbf{R}^3 \times \mathbf{R}^3} dw (1 + |w|^2 + |W|^2) M(W) \Pi_{gp}^{0,0}(w, dV dW) \in L^1((1 + V^2)^{-3} dV).$$

IPOTESI SU Π_{pg}, Π_{gp} -STOKES

Ipotesi (H4): Se $(\epsilon, \eta) \rightarrow (0, 0)$, $\Pi_{gp}^{\epsilon, \eta}(w, \cdot) \rightarrow \Pi_{gp}^{0,0}(w, \cdot)$ debole nel senso delle misure di prob. per q.o. $w \in \mathbf{R}^3$, con

$$\mathcal{T}_R \# \Pi_{gp}^{0,0} = \Pi_{gp}^{0,0} \quad \text{per ogni } R \in O_3(\mathbf{R}), \quad (22)$$

Inoltre, per ogni $p > 3$ e ogni $\Phi := \Phi(w, W) \in C^1(\mathbf{R}^3 \times \mathbf{R}^3)$ t.c.

$$|\Phi(w, W)| + |\nabla_w \Phi(w, W)| \leq C(1 + |w|^2 + |W|^2)M(W),$$

per $C > 0$, si ha

$$\begin{aligned} & \int_{\mathbf{R}^3} (1 + |V|^2)^{-p} \left| \int_{\mathbf{R}^3 \times \mathbf{R}^3} \Phi(w, W) (\Pi_{gp}^{\epsilon, \eta}(w, dV dW) dw - \Pi_{gp}^{0,0}(w, dV dW) dw) \right| \\ & \quad = O(\epsilon + \eta) \\ & \quad \text{se } (\epsilon, \eta) \rightarrow 0. \end{aligned}$$

IPOTESI SU Π_{pg}, Π_{gp}

Ipotesi (H5). Per ogni $h \in L^2(M(w)dw)$,

$$\int_{\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3} (1 + |W|^2)(1 + |V|^2)^{-p}(1 + |w|^2)M(W)|h(W)|\Pi_{gp}^{\epsilon, \eta}(w, dVdW) dw \leq C \|h\|_{L^2(M(w)dw)},$$

con $C > 0$ indipendente da η e ϵ (per η e ϵ piccoli)

MODELLO BOLTZMANN MULTIFASE

LE PRECEDENTI IPOTESI INCLUDONO I NUCLEI CORRISPONDENTI A:

- Collisioni Elastiche

MODELLO BOLTZMANN MULTIFASE

LE PRECEDENTI IPOTESI INCLUDONO I NUCLEI CORRISPONDENTI A:

- Collisioni Elastiche
- Alcuni tipi di collisioni anelastiche e.g. F. Charles: in "Proceedings of the 26th International Symposium on Rarefied Gas Dynamics", AIP Conf. Proc. 1084, (2008), 409–414
(Collisioni con riflessione diffusa)

Si assume poi

$$f(t, x, w) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{w^2}{2}} (1 + \epsilon g(t, x, w)) = M(1 + \epsilon g(t, x, w))$$

(**perturbazione intorno a Maxwelliana locale**)
e si considera l'integrale di collisione linearizzato

$$\mathcal{L}\phi := -M^{-1}DC(M) \cdot (M\phi), \quad (23)$$

(D derivata funzionale).

che verifica

- $\mathcal{L}$ è un operatore non limitato su $L^2(Mdv)$ con
 $\text{Dom } \mathcal{L} = L^2((\bar{c} \star M)^2 Mdv)$, dove $\bar{c}(z) := \int_{\mathbb{S}^2} c(z, \omega) d\omega$
- $\mathcal{L} = \mathcal{L}^* \geq 0$ con $\text{Ker } \mathcal{L} = \text{Span}\{1, w_1, w_2, w_3, |w|^2\}$
- $\mathcal{L}$ Fredholm, $\text{Im } \mathcal{L} = \text{Ker } \mathcal{L}^\perp$.

$$A(w) := w \otimes w - \frac{1}{3}|w|^2 I \quad (A \perp \text{Ker } \mathcal{L} \text{ in } L^2(M dv)) \quad (24)$$

$\implies$

$\exists! \tilde{A} \in \text{Dom } \mathcal{L} \text{ t.c.}$

$$\mathcal{L} \tilde{A} = A, \quad \tilde{A} \perp \text{Ker } \mathcal{L}. \quad (25)$$

Proprietà di simmetria dell'integrale di collisione + invarianza per rotazione della Maxwelliana $\implies$

$$\tilde{A}(w) = \alpha(|w|)A(w), \quad (26)$$

con α misurabile t.c.

$$\int_{\mathbf{R}^3} \alpha(|w|)^2 |w|^4 (\bar{c} \star M(w))^2 M(w) dw < \infty.$$

IPOTESI SU c

Ipotesi (H6). $c \equiv c(z, \omega)$ è della forma

$$c(w - w_*, \omega) = |w - w_*| \sigma_{gg}(|w - w_*|, |\cos(\widehat{w - w_*}, \omega)|), \quad (27)$$

e soddisfa

$$0 < c(z, \omega) \leq c_*(1 + |z|)^\gamma, \quad \text{per q.o. } (z, \omega) \in \mathbf{R}^3 \times \mathbf{S}^2,$$

$$\int_{\mathbf{S}^2} c(z, \omega) d\omega \geq \frac{1}{c_*} \frac{|z|}{1 + |z|}, \quad \text{per q.o. } z \in \mathbf{R}^3,$$

per qualche $c_* > 1$ e $\gamma \in [0, 1]$.

IPOTESI SU c -NAVIER-STOKES

Ipotesi (H7). La funzione $\alpha \equiv \alpha(|w|)$ t.c.

$$\iint_{\mathbf{R}^3 \times \mathbf{S}^2} (\alpha(|w|)w_1w_2 + \alpha(|w_*|)w_{*1}w_{*2} - \alpha(|w'|)w'_1w'_2 - \alpha(|w'_*|)w'_{*1}w'_{*2}) c(w - w_*, \omega) M(w_*) dw_* d\omega = w_1w_2$$

è t.c. $\alpha \in L^\infty(\mathbf{R}_+)$.

Tale condizione definisce un'unica funzione radiale (a meno di un insieme a misura nulla) $\alpha \in L^\infty(\mathbf{R}_+)$ t.c.

$$\mathcal{L}(\alpha A_{kl}) = A_{kl} \quad \text{for all } k, l = 1, 2, 3.$$

(L. Desvillettes, F. Golse, in "Advances in Kinetic Theory and Computing", Series on Advances in Mathematics for Applied Sciences, Vol. 22, (1994), 191–203.)

IPOTESI SU c -STOKES

Ipotesi H7: $\exists C > 0$ tale che

$$|\tilde{A}(w)| \leq C(1 + |w|^2)$$

e

$$|\nabla \tilde{A}(w)| \leq C(1 + |w|^2) .$$

Teorema (NS) Si considerino $\Pi_{pg}^{\epsilon_n, \eta_n}$ e $\Pi_{gp}^{\epsilon_n, \eta_n}$ verificanti (H1)-(H5), c verificante (H6)-(H7), e lo scaling (Vlasov-Navier-Stokes):

$$\epsilon_n \rightarrow 0, \quad \text{e} \quad \eta_n / \epsilon_n^2 \rightarrow 0. \quad (28)$$

Siano $g_n \equiv g_n(t, x, w) \geq 0$ e $F_n \equiv F_n(t, x, v) \geq 0$ successioni di funzioni regolari (almeno C^1) t.c.

$F_n \rightharpoonup F$ in L_{loc}^∞ deb-*, e $g_n \rightharpoonup g$ in $L_{loc}^2(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$ deb
con $F \in L_{loc}^\infty(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$, $g \in L_{loc}^2(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$, e

$$f_n(t, x, w) := M(w)(1 + \epsilon_n g_n(t, x, w)), \quad (29)$$

TEOREMA NS

Si assuma che

(a) (F_n, f_n) risolve (VNS)

(b) $\exists p > 3$ tale che, $\forall R > 0$,

$$\sup_{n \geq 1} \sup_{(t,x,v) \in [0,R] \times [-R,R]^3 \times \mathbf{R}^3} (1 + |v|^2)^p F_n(t, x, v) \leq C_R < \infty$$

(c) $\int_{\mathbf{R}^3} g_n(t, x, w)^2 M(w) dw$ limitato in $L^1_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^3)$,

(d) $\int_{\mathbf{R}^3} g_n \phi(w) M(w) dw \rightarrow \int_{\mathbf{R}^3} g \phi(w) M(w) dw$ forte in $L^2_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^3)$ per ogni $\phi \in C_c(\mathbf{R}^3)$.

TEOREMA NS

Allora, $\exists$ le funzioni L^∞ $\rho \equiv \rho(t, x) \in \mathbf{R}$, $\theta \equiv \theta(t, x) \in \mathbf{R}$,
 $u \equiv u(t, x) \in \mathbf{R}^3$ t.c., per q.o. $(t, x, w) \in \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$,

$$g(t, x, w) = \rho(t, x) + u(t, x) \cdot w + \theta(t, x) \frac{1}{2}(|w|^2 - 3)$$

e (F, u) soddisfa, nel senso delle distribuzioni,

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = \kappa \operatorname{div}_v((v - u)F), \\ \operatorname{div}_x u = 0, \\ \partial_t u + \operatorname{div}_x(u \otimes u) = \nu \Delta_x u - \nabla_x p + \kappa \int (v - u)F \, dv, \end{cases} \quad (30)$$

con

$$\nu := \frac{1}{10} \int \tilde{A} : \mathcal{L} \tilde{A} M(w) \, dw > 0, \quad \kappa := \frac{1}{3} \int Q(|w|)|w|^2 M(w) \, dw > 0. \quad (31)$$

Teorema S Siano $g_n \equiv g_n(t, x, w)$ e $F_n \equiv F_n(t, x, v) \geq 0$ successioni di funzioni regolari (almeno C^1). Si assuma che F_n e f_n definita da

$$f_n(t, x, w) = M(w)(1 + \epsilon_n g_n(t, x, w)), \quad (32)$$

siano soluzioni del sistema (VS), che $\Pi_{pg}^{\epsilon_n, \eta_n}$ e $\Pi_{gp}^{\epsilon_n, \eta_n}$ soddisfino (H1)-(H5) e che l'interazione tra molecole soddisfi (H1) e (H2).

Si consideri lo scaling

$\epsilon_n \rightarrow 0$, $\eta_n / \epsilon_n^2 \rightarrow 0$, $\epsilon_n / \mu_n^2 \rightarrow 0$, $\mu_n \rightarrow 0$, e si assuma

che $F_n \rightharpoonup F$ in $L_{loc}^\infty(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$ deb-*, e

$g_n \rightharpoonup g$ in $L_{loc}^2(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$ deb.

TEOREMA S

Si assuma che

(a) F_n è t.c., per qualche $p > 3$,

$$\sup_{n \geq 1} \sup_{t, |x| \leq R} \sup_{v \in \mathbf{R}^3} (1 + |v|^2)^p F_n(t, x, v) < \infty$$

$\forall R > 0$.

(b) per qualche $q > 1$,

$$\sup_{t, |x| < R} \int_{\mathbf{R}^3} (1 + |w|^2)^q M(w) g_n^2(t, x, w) dw < \infty$$

$\forall R > 0$.

TEOREMA S

Allora $\exists \rho \equiv \rho(t, x) \in \mathbf{R}, \theta \equiv \theta(t, x) \in \mathbf{R}, u \equiv u(t, x) \in \mathbf{R}^3$ funzioni L^∞ t.c. per q.o. $t, x \in \mathbf{R}_+^* \times \mathbf{R}^3$,

$$g(t, x, w) = \rho(t, x) + u(t, x) \cdot w + \theta(t, x) \frac{1}{2}(|w|^2 - 3), \quad (33)$$

con u, F soluzione, nel senso delle distribuzioni, di

$$\left\{ \begin{array}{l} \partial_t F + v \cdot \nabla_x F = \kappa \operatorname{div}_v((v - u)F), \\ \operatorname{div}_x u = 0, \\ -\nu \Delta_x u + \nabla_x p = \kappa \int_{\mathbf{R}^3} (v - u)F \, dv, \end{array} \right. \quad (34)$$

$$\mathbf{e} \, \nu := \frac{1}{10} \int \tilde{A} : \mathcal{L} \tilde{A} M \, dw > 0, \quad \kappa := \frac{1}{3} \int Q(|w|) |w|^2 M \, dw > 0.$$

Schema della dimostrazione

Deflessione

$$\frac{1}{\eta} \mathcal{D}(F_\epsilon, f_\epsilon) \rightarrow \kappa \operatorname{div}_v((v - u)F(v)) \text{ in } \mathcal{D}'_{t,x}$$

in particolare, con calcolo esplicito,

$$\begin{aligned} \frac{1}{\eta_n} \int_{\mathbf{R}^3} \mathcal{D}(F_n, f_n)(v) \phi(v) \, dv &= -\kappa \int_{\mathbf{R}^3} F(V) \nabla \phi(V) \cdot (V - u) \, dV \\ &\quad + O(\epsilon_n) + O\left(\frac{\eta_n}{\epsilon_n^2}\right) \end{aligned}$$

La dimostrazione è comune per i due scaling ed usa le ipotesi (H1), (H2), (H3) e (b), (c), (d) del teorema NS (o (a) e (b) del teorema S).

Attrito

$$\frac{1}{\epsilon} \int w \mathcal{R}(f_\epsilon, F_\epsilon)(w) dw \rightarrow \kappa \int (v - u) F(v) dv \text{ in } \mathcal{D}'_{t,x}$$

Usando (H1),(H2) e il risultato precedente:

$$\begin{aligned} \frac{1}{\epsilon_n} \int_{\mathbf{R}^3} w \mathcal{R}(f_n, F_n) dw &= \frac{1}{\epsilon_n} \iiint (w - W) f_n(W) F_n(V) \Pi_{gp}(w, dV dW) dw \\ &= -\frac{1}{\eta_n} \iiint (v - V) F_n(V) f_n(W) \Pi_{pg}(v, dV dW) dv = -\frac{1}{\eta_n} \int_{\mathbf{R}^3} v \mathcal{D}(F_n, f_n) dv \\ &\rightarrow \kappa \int_{\mathbf{R}^3} F(V) (V - u) dV \end{aligned}$$

$$(\phi(v) = v).$$

MODELLO BOLTZMANN MULTIFASE

Per i calcoli successivi, l'equazione cinetica può essere riscritta come

$$\partial_t g_\epsilon + \frac{1}{\epsilon} w \cdot \nabla_x g_\epsilon = -\frac{1}{\mu\epsilon} \mathcal{R}(M(1 + \epsilon g_\epsilon), F_\epsilon) - \frac{\mu}{\epsilon^2} \mathcal{L}(g_\epsilon) + \frac{\mu}{\epsilon} Q(g_\epsilon, g_\epsilon)$$

con

$$\mathcal{L}(g) = -M^{-1} DC(M)Mg$$

e

$$Q(g, g) = -M^{-1} \mathcal{C}(Mg).$$

Termine dominante nell'equazione per le molecole

$\mathcal{L}(g_\epsilon) \rightarrow \mathcal{L}g = 0$, condizione che fissa g , che deve essere nel nucleo

$$g(t, x, w) = \rho(t, x) + u(t, x) \cdot w + \theta(t, x) \frac{1}{2}(|w|^2 - 3)$$

Per dimostrare la convergenza, si usano le ipotesi (H1) e e (b), (c) del teorema NS (o (a) e (b) del teorema S)

Incompressibilità

Dalla conservazione locale della massa

$$\epsilon \partial_t \int M(w) g_\epsilon dw + \operatorname{div}_x \int M(w) w g_\epsilon dw = 0$$

si ottiene $\implies \operatorname{div}_x u = 0$

Per completare la prova (C.Bardos-F.Golse-C.D.Levermore J.Stat.Phys.**63**, 323–344 (1991)), si ricorda che, dato che $\mathcal{L}$ è autoaggiunto in $L^2(M \, dw)$,

$$\frac{\mu_n}{\epsilon_n} \int M A(w) g_n = \int M \tilde{A}(w) \frac{\mu_n}{\epsilon_n} \mathcal{L} g_n .$$

e quindi si può scrivere

$$\begin{aligned} \frac{\mu_n}{\epsilon_n} \int M A(w) g_n &= \mu_n \int M \tilde{A}(w) \mathcal{Q}(g_n) - \int M \tilde{A}(w) (\epsilon_n \partial_t + w \cdot \nabla_x) g_n \\ &\quad + \frac{1}{\mu_n} \int M \tilde{A}(w) M^{-1} \mathcal{R}(f_n, F_n). \end{aligned}$$

per i quali si hanno i seguenti andamenti:

$$\int M \tilde{A}(w) (\epsilon_n \partial_t + w \cdot \nabla_x) g_n \rightarrow \nu (\nabla_x u + (\nabla_x u)^T) \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3).$$

$\mu_n \int M \tilde{A}(w) \mathcal{Q}(g_n)$ converge a 0 nell'asintotica S e a $A(u)$ in quella NS

e

$$\frac{1}{\mu_n} \int M \tilde{A}(w) M^{-1} \mathcal{R}(f_n, F_n) \rightarrow 0 \text{ nelle due asintotiche}$$

quindi

$$\begin{aligned} \operatorname{div}_x \frac{\mu_n}{\epsilon_n} \langle A(w) g_n \rangle &\rightarrow (\lim_n \mu_n) \left(\operatorname{div}_x (u \otimes u) - \frac{1}{3} \nabla_x |u|^2 \right) - \nu \Delta_x u - \nu \nabla_x \operatorname{div}_x u \\ &= (\lim_n \mu_n) \left(\operatorname{div}_x (u \otimes u) - \frac{1}{3} \nabla_x |u|^2 \right) - \nu \Delta_x u \end{aligned}$$

in $\mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3)$.

L'equazione di conservazione del momento può esser riscritta opportunamente come

$$\mu_n \partial_t \int M w g_n + \frac{\mu_n}{\epsilon_n} \operatorname{div}_x \int M w^{\otimes 2} g_n = \frac{1}{\epsilon_n} \int M w M^{-1} \mathcal{R}(f_n, F_n) .$$

che asintoticamente diventa

$$(\lim_n \mu_n) \partial_t u + \operatorname{div}_x ((\lim_n \mu_n) u \otimes u - \nu \nabla_x u) - \kappa \int_{\mathbf{R}^3} (v - u) F \, dv = -\nabla_x p .$$

L'equazione limite per F si ottiene invece dalle proprietà di convergenza e dal calcolo esplicito del limite del termine di deflessione, ed è

$$\partial_t F + v \cdot \nabla_x F = \kappa \operatorname{div}_v ((v - u) F) . \quad \blacksquare$$

CONCLUSIONE

- Sono stati analizzati modelli "microscopici" per il sistema di EDP macroscopiche

$$\left\{ \begin{array}{l} \partial_t F + v \cdot \nabla_x F - \kappa \operatorname{div}_v [(v - U)nF] = 0 \\ (\partial_t U + U \cdot \nabla_x U) - \nu \Delta U + \nabla \Pi = (g) + \kappa (j - \rho U), \\ \nabla \cdot U = 0, \end{array} \right.$$

legato (per es.) allo studio di spray sottili/aerosol.

- La trattazione microscopica a scale inferiori a quella mesoscopica presenta difficoltà di difficile soluzione.
- È stata data una derivazione del sistema completo scegliendo come microscala la scala mesoscopica, per una classe generale di nuclei di collisione particelle-molecole, che include le collisioni elastiche e alcuni tipi di collisioni anelastiche



Costruzione di un modello di Maxwell-Vlasov per la propagazione e l'assorbimento di onde elettromagnetiche in plasmi confinati magneticamente.

Alessandro Cardinali

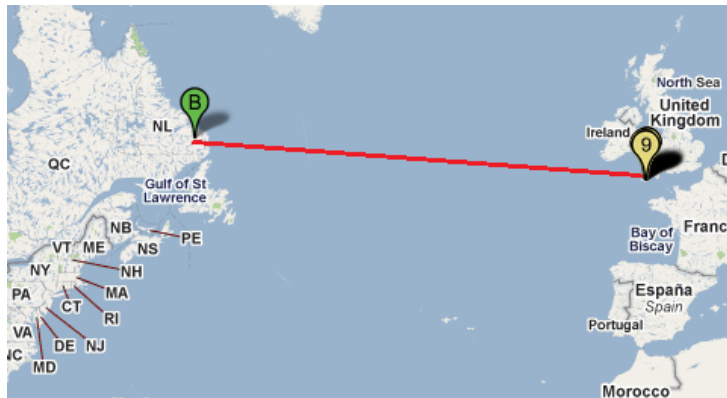
ENEA, Dipartimento Fusione e Sicurezza Nucleare,
Divisione Fisica della Fusione,
Centro Ricerche Frascati,
Via E. Fermi 45, 00044 Frascati (Roma), Italy

CNR
Roma, 6 giugno 2018

Summary

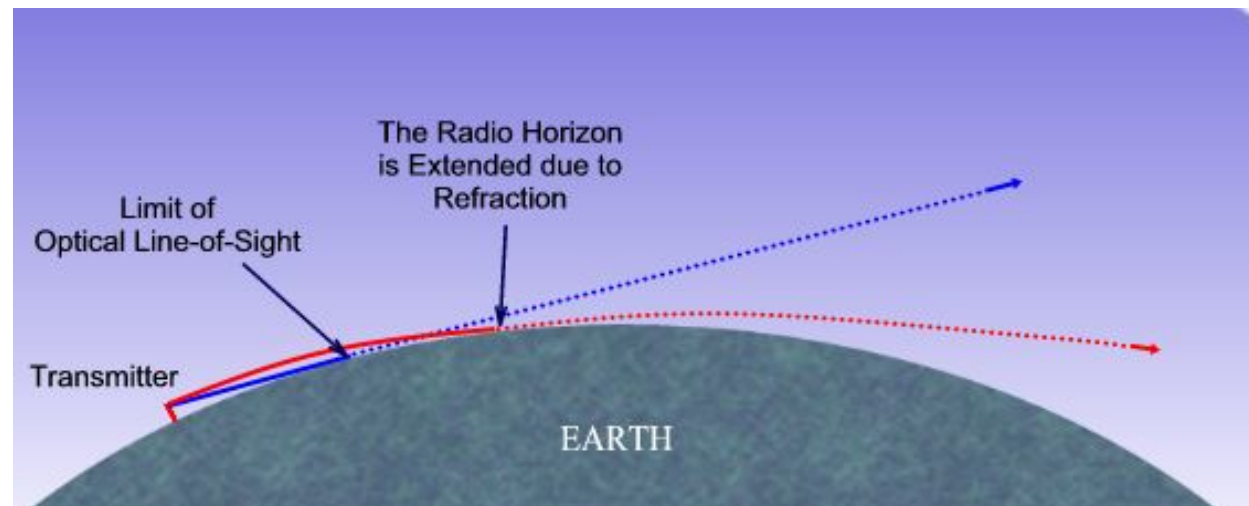
- Definition of plasma (particles and waves),
- Maxwell's equations in a magnetized plasma
 - Meaning
 - Kinetic approach (notes)
 - Fluid Approach
- Simplifying Hypothesis and Fourier analysis: dispersion relation
 - General thermal dispersion relation
 - Cold limit
- Wave equation for a cold plasma and harmonic field
- The case of the Lower Hybrid Wave (LHW)
- Solution of the wave equation in the case of bounded and homogeneous plasma
 - Hint to Asymptotic Methods and WKB Approximation for the Lower Hybrid waves
 - Numeric results
- Conclusions

Historical excursus.....



Marconi's experiment (1901, Nobel prize 1904)

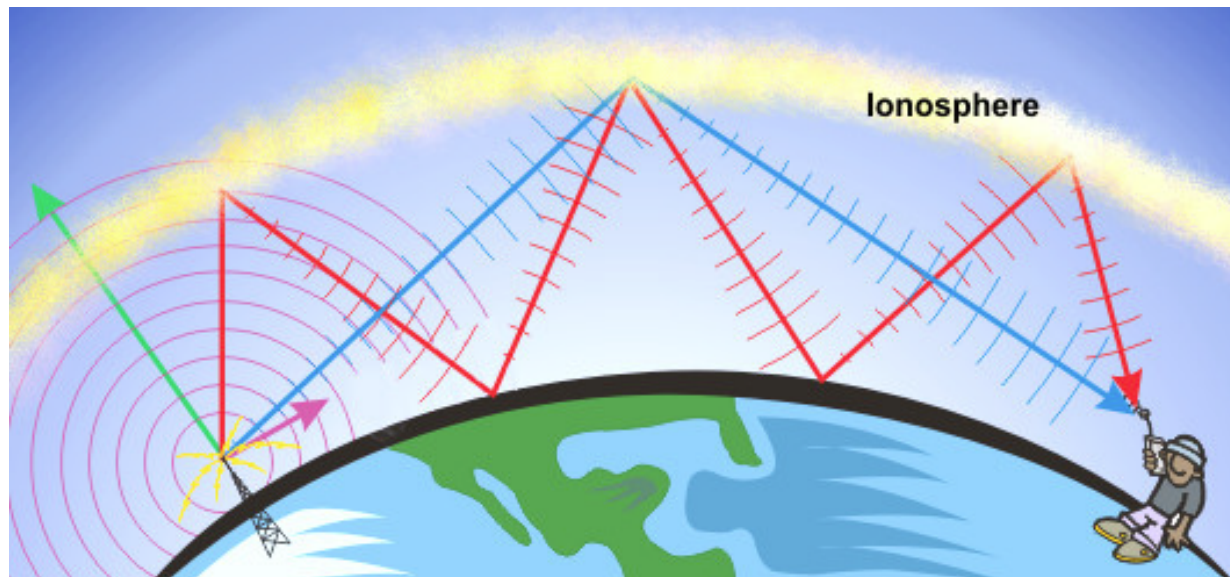
Why does not the ray follow a straight path?



R. L. Stenzel, Whistler waves in space and laboratory plasmas JGR, 104 A7 (1999)

D. A. Gurnett, Waves in space plasma: A historical overview of the sounds of space, GRA 8 03591 (2006)

E. V. Appleton, M. Barnett & G. Breit, M. Tuve in 1924 showed the existence in the atmosphere of a layer (called ionosphere (60-500Km)) with the characteristics of an ionized gas (*Appleton: Nobel prize 1947*).



The propagation of the em waves (in particular RF waves) in the ionosphere is different from that in the air or in vacuum. Sometime the signal can reflect in some layers and remain trapped between the ground and the reflection layer with the possibility to reach all the points on the earth surface.

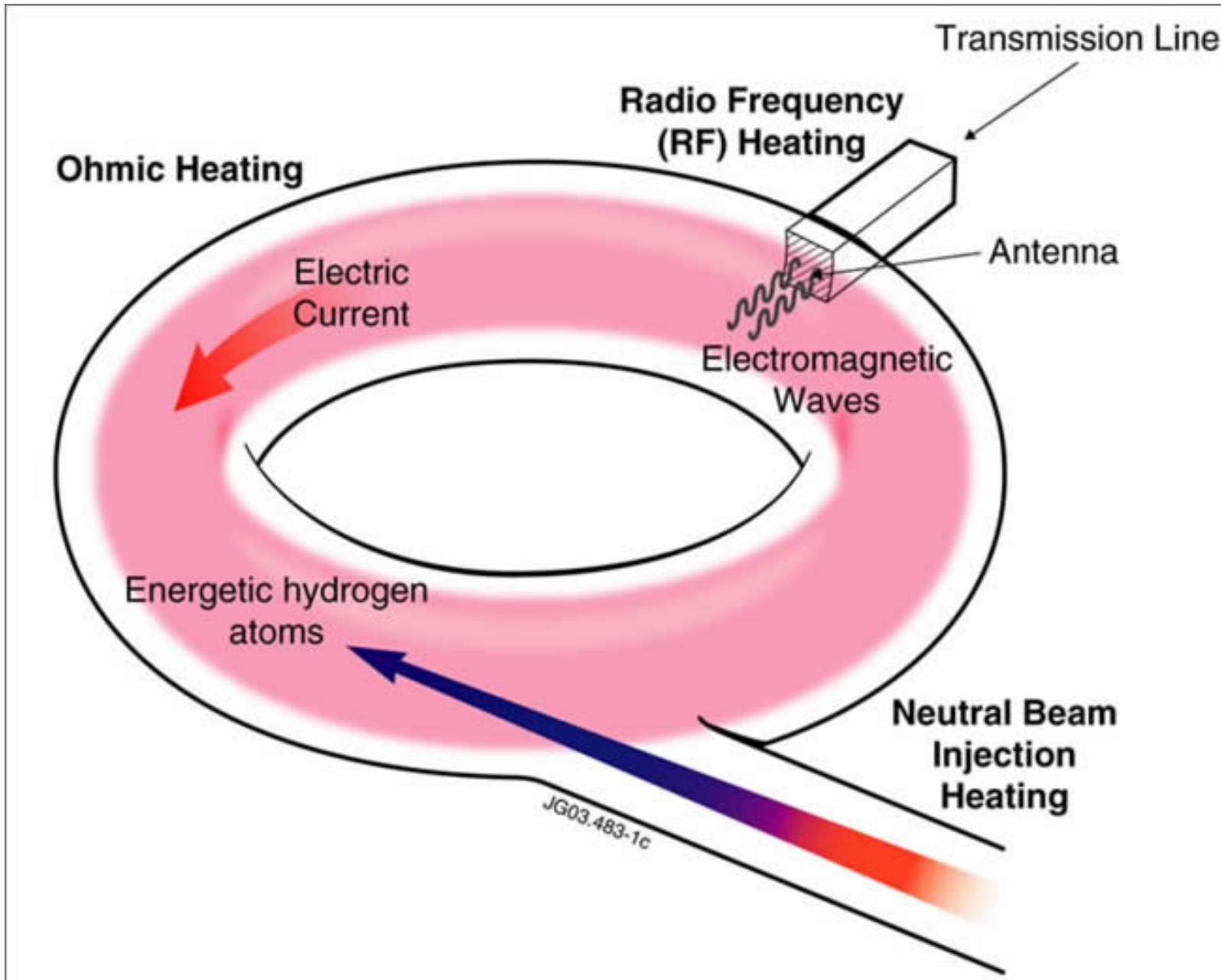
- In the 1960s and 1970s, the interest in the study of propagation and absorption of radio waves in ionized (or plasma) gases was also directed in the field of research on the control of thermonuclear fusion in plasma confined magnetically in machines called "tokamak".
- The so-called plasma heating through high-power radio waves

Frequencies range for application to Nuclear Fusion

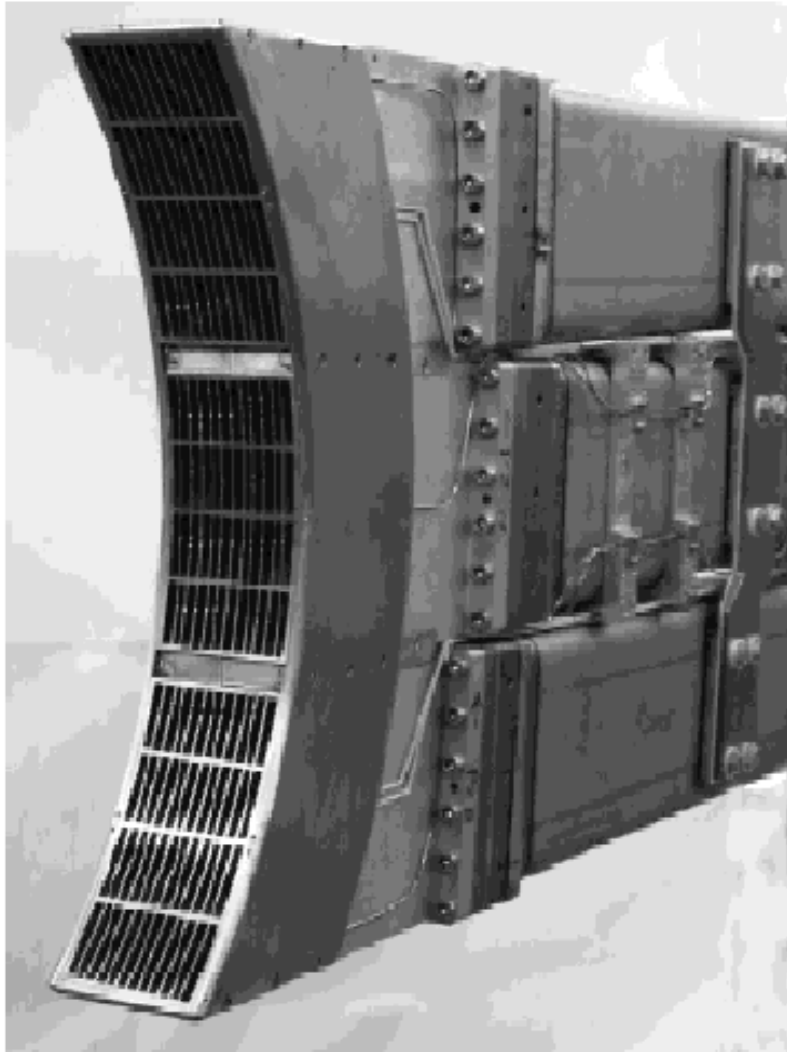
Table 1: Frequency Band Designations

f	λ	Band	Description
30–300 Hz	10^4 – 10^3 km	ELF	Extremely low frequency
300–3000 Hz	10^3 – 10^2 km	VF	Voice frequency
3–30 kHz	100–10 km	VLF	Very low frequency
30–300 kHz	10–1 km	LF	Low frequency
0.3–3 MHz	1–0.1 km	MF	Medium frequency
3–30 MHz	100–10 m	HF	High frequency
30–300 MHz	10–1 m	VHF	Very high frequency
300–3000 MHz	100–10 cm	UHF	Ultra-high frequency
3–30 GHz	10–1 cm	SHF	Superhigh frequency
30–300 GHz	10–1 mm	EHF	Extremely high frequency (millimeter waves)

Sketch of a toroidal device oriented to the Control of Nuclear Fusion



Photography of the antenna for RF injection in the Tokamak-plasma of FTU



- Electromagnetic waves at 8GHz and 3MW are injected into FTU plasma to increase the plasma temperature. The wave energy is transferred to the plasma through the "Landau Damping" mechanism.
- Moreover, the RF can induce a toroidal current that can replace the inductive current and allow steady-state operations.

- Before presenting the mathematical model that allows the description of the em waves in the plasma, we refer to some concepts and definitions
 - Plasma is said to be a system whose dynamics is dominated by electromagnetic forces: plasma is the set of charged particles and the fields they generate
 - A necessary condition for a system of charged particles to be a plasma is that the number of particles in the Debye sphere must be very large

$$n\lambda_D^3 = 1.3 \times 10^6 T_{keV}^{1/2} n_{e10^{14} cm^{-3}}^{-1/2} \gg 1$$

A plasma has two fundamental properties:

- *i) is able of conducting an electrical current*
- *ii) Interacts with electromagnetic fields both in the sense*
 - *of being able to actively generate it (as well as any moving electric charge)*
 - *passively, in the sense of undergoing the effects of electromagnetic fields applied from the outside*

Maxwell's equations in plasma



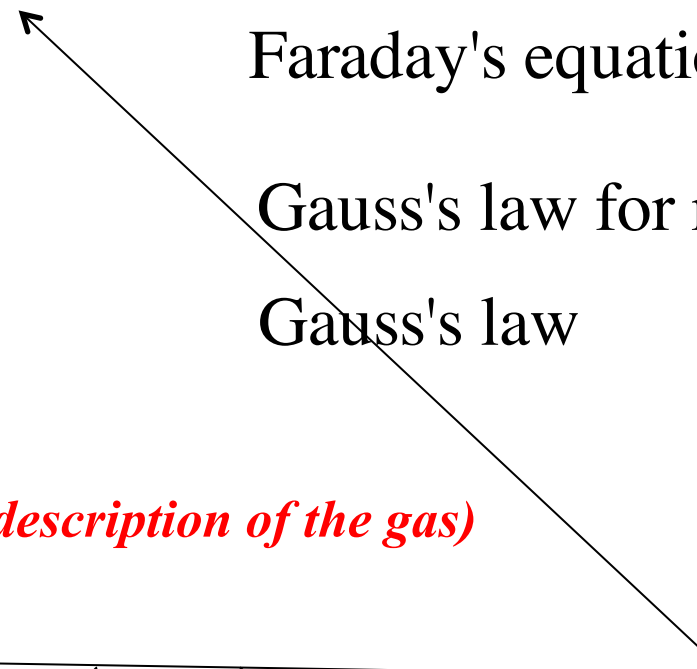
1) the Maxwell's equation system (for the description of the em fields)

$$\nabla \wedge \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{J} \Rightarrow \text{Ampere's equation}$$

$$\nabla \wedge \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \Rightarrow \text{Faraday's equation}$$

$$\nabla \cdot \vec{B} = 0 \Rightarrow \text{Gauss's law for magnetism}$$

$$\nabla \cdot \vec{D} = 4\pi\rho \Rightarrow \text{Gauss's law}$$



2) "Boltzmann's equation" (statistical description of the gas)

Forza di Lorentz

$$\partial_t f_\alpha + \vec{v} \cdot \nabla f_\alpha + \frac{\vec{F}}{m_\alpha} \cdot \nabla_{\vec{v}} f_\alpha = (\delta_t f_\alpha)_{coll} \quad \vec{F} = q_\alpha \left(\vec{E} + \frac{\vec{v} \wedge \vec{B}}{c} \right)$$

Simplification of the Maxwell's equations

- Based on the plasma definition, the following simplifications can be made to the Maxwell-Boltzmann system in plasmas
 - Permeability $\mu=1$ (the same as in vacuum) $B=H$
 - No distinction between polarization and conduction current and between free and polarization charge

$$\nabla \wedge \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} (\vec{J}_p + \vec{J}_{ext}) \Rightarrow \text{Ampere's equation}$$

$$\nabla \wedge \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \Rightarrow \text{Faraday's equation}$$

$$\vec{J} = \vec{J}_{ext} + \vec{J}_p = \vec{J}_{ext} + q_\alpha \int \vec{v} f_\alpha(\vec{v}, \vec{r}, t) d\vec{v}$$

$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot \vec{J}_p = 0$$

The Gauss laws can immediately be deduced from the above equations and conservation of the charge

$$\nabla \wedge \nabla \wedge \vec{E} = \left[-\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi q_\alpha}{c^2} \frac{\partial}{\partial t} \left(\int_{-\infty}^{+\infty} \vec{v} f_\alpha(\vec{r}, \vec{v}, t, \vec{E}(\vec{r}, t)) d\vec{v} + \vec{J}^{ext} \right) \right]$$

$$\partial_t f_\alpha + \vec{v} \cdot \nabla f_\alpha + \frac{q}{m_\alpha} \left(\vec{E} + \frac{\vec{v} \wedge \vec{B}_{ext}}{c} \right) \cdot \nabla_{\vec{v}} f_\alpha = (\delta_t f_\alpha)_{coll}$$

Supported by initial-boundary conditions on the plasma surface

This is a non-linear integro-differential equation system. No way to extract information from this system. Impossibility to find a solution

$$\vec{E}^m(\vec{r}, t) = \vec{E}(\vec{r}, t) + \delta \vec{E}(\vec{r}, t) \Rightarrow \langle \delta \vec{E}(\vec{r}, t) \rangle = 0$$

- The solution of the Maxwell-Vlasov system is practically impossible even in simple cases
- How to extract information from the Maxwell-Vlasov equation system?
 - From integro-differential to algebraic equation
 - *Hypothesis about the plasma*
 - Stationary
 - Homogeneous
 - Unbounded
 - Neglecting collisions
 - Weakness of the electromagnetic perturbation
 - Linearization of the Vlasov equation $(W_{\text{wave}} \equiv |E|^2 \ll W_{\text{thermal}} \equiv \kappa T)$
 - Linear correlation between the current and the electric field $\vec{J}_{\text{pol}}(\vec{r}, t) \sim \mathcal{Q}(\vec{E}(\vec{r}, t))$
 - Maxwellian background for the equilibrium of the distribution function
 - This allows to apply the Fourier transform for the perturbed quantities (fields and perturbed distribution

$$\tilde{\vec{E}}(\omega, \vec{k}) = \int dt \int d\vec{r} \vec{E}(\vec{r}, t) e^{-i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\tilde{f}_1(\omega, \vec{k}) = \int dt \int d\vec{r} f_1(\vec{r}, t) e^{-i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\nabla \wedge \nabla \wedge \vec{E} = \left[-\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi q_\alpha}{c^2} \frac{\partial}{\partial t} \left(\int_{-\infty}^{+\infty} \vec{v} f_\alpha(\vec{r}, \vec{v}, t, \vec{E}(\vec{r}, t)) d\vec{v} + \vec{J}^{ext} \right) \right]$$
$$\partial_t f_\alpha + \vec{v} \cdot \nabla f_\alpha + \frac{q}{m_\alpha} \left(\vec{E} + \frac{\vec{v} \wedge \vec{B}_{ext}}{c} \right) \cdot \nabla_{\vec{v}} f_\alpha = (\delta_t f_\alpha)_{coll}$$

The equation system above transform in a system of algebraic equations correlating omega to wave-vector

$$\vec{k} \wedge \vec{k} \wedge \vec{\tilde{E}}_{\vec{k}, \omega} + \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}}(\vec{k}, \omega) \cdot \vec{\tilde{E}}_{\vec{k}, \omega} = 0$$

In order to have no trivial solutions

$$\det \left| k^2 (\kappa_i \kappa_j - \delta_{ij}) + \frac{\omega^2}{c^2} \varepsilon_{ij}(\vec{k}, \omega) \right| = 0$$

Dispersion Relation

$$\underline{\underline{\epsilon}}(\vec{k}, \omega) = \left[\underline{\underline{I}} + \frac{4\pi i q_{\alpha}}{\omega} \underline{\underline{\sigma}}(\vec{k}, \omega) \right]$$

➤ Properties

- holomorphic function of omega and k
- Decomposition in Hermitian and anti-Hermitian parts
- The real part of the Hermitian and the imaginary part of the anti-Hermitian are symmetric
- The imaginary part of the Hermitian and the real part of the anti-Hermitian are anti-symmetric

$$\underline{\underline{\epsilon}}^H(\vec{k}, \omega) = \frac{1}{2} \left(\underline{\underline{\epsilon}}(\vec{k}, \omega) + \underline{\underline{\epsilon}}^*(\vec{k}, \omega) \right)$$

$$\underline{\underline{\epsilon}}^A(\vec{k}, \omega) = \frac{1}{2i} \left(\underline{\underline{\epsilon}}(\vec{k}, \omega) - \underline{\underline{\epsilon}}^*(\vec{k}, \omega) \right)$$

The dielectric tensor after laborious calculations can be written as

$$\epsilon_{ij} = \delta_{ij} - \sum_{\alpha} \Pi_{ij}^{\alpha} \sum_{n=-\infty} \Gamma_{ij}^n(\lambda_{\alpha}) Z_{ij}(x_{n\alpha})$$

For example, the component xx is written as

$$\epsilon_{xx} = 1 - \sum_{\alpha} \Pi_{xx}^{\alpha} \sum_{n=-\infty}^{+\infty} \Gamma_{xx}^n(\lambda_{\alpha}) Z_{xx}(x_{n\alpha})$$

$$\Pi_{xx}^{\alpha} = \frac{\omega_{p\alpha}^2}{\omega^2}$$

$$\Gamma_{xx}^n(\lambda_{\alpha}) = \frac{n^2}{\lambda_{\alpha}} I_n(\lambda_{\alpha}) e^{-\lambda_{\alpha}}$$

$$Z_{xx}(x_{n\alpha}) = -x_{0\alpha} Z(x_{n\alpha})$$

$$x_{n\alpha} = \frac{\omega - n\Omega_{c\alpha}}{k_{\parallel} v_{th\alpha}}$$

$$\lambda_{\alpha} = \frac{k_{\perp}^2 v_{th\alpha}^2}{2\Omega_{c\alpha}^2} = k_{\perp}^2 \rho_{\alpha}^2$$

$$Z'(x_{n\alpha}) = -2[1 + x_{n\alpha} Z(x_{n\alpha})]$$

The cold limit $T \rightarrow 0$

$$\underline{\underline{\varepsilon}}^A(\omega) \rightarrow 0$$

$$\underline{\underline{\varepsilon}}^H(\omega) = S \underline{\underline{I}} + (P + S) \vec{b} \vec{b} - iD \underline{\underline{\varepsilon}}^{LC} \cdot \vec{b}$$

$$\varepsilon_{xx} = S = \frac{R+L}{2} = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_{c\alpha}^2}$$

$$\varepsilon_{xy} = -iD = \frac{R-L}{2i} = -i \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2 - \Omega_{c\alpha}^2} \frac{\Omega_{c\alpha}}{\omega}$$

$$\varepsilon_{zz} = P = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2}$$

$$R = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} \frac{\omega}{\omega + \Omega_{c\alpha}}$$

$$L = 1 - \sum_{\alpha} \frac{\omega_{p\alpha}^2}{\omega^2} \frac{\omega}{\omega - \Omega_{c\alpha}}$$

$$\vec{b} = \vec{B} / |\vec{B}|$$

$$\underline{\underline{\varepsilon}}^{LC} \text{ is the Levi-Civita symbol of rank 3}$$

The cold dispersion relation

$$\det \left| k^2 (\kappa_i \kappa_j - \delta_{ij}) + \frac{\omega^2}{c^2} \varepsilon_{ij}(\vec{k}, \omega) \right| = 0 \Rightarrow Ak^4 + Bk^2 + C = 0$$

$$A = S \sin^2 \theta + P \cos^2 \theta$$

$$B = (S^2 - D^2) \sin^2 \theta + PS(1 + \cos^2 \theta) = RL \sin^2 \theta + PS(1 + \cos^2 \theta)$$

$$C = P(S^2 - D^2) = PRL$$

Theta is the angle between k and B

$$\nabla \wedge \nabla \wedge \vec{E} = \left[-\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi q_\alpha}{c^2} \frac{\partial}{\partial t} \left(\int_{-\infty}^{+\infty} \vec{v} f_\alpha(\vec{r}, \vec{v}, t, \vec{E}(\vec{r}, t)) d\vec{v} + \vec{J}^{ext} \right) \right]$$
$$\partial_t f_\alpha + \vec{v} \cdot \nabla f_\alpha + \frac{q}{m_\alpha} \left(\vec{E} + \frac{\vec{v} \wedge \vec{B}_{ext}}{c} \right) \cdot \nabla_{\vec{v}} f_\alpha = (\delta_t f_\alpha)_{coll}$$

Goal: *Obtain a Differential equation system in space*

- **Maintain**
 - Stationary
 - Neglecting collisions
 - Weakness of the electromagnetic perturbation
 - Linearization of the Vlasov's equation
 - Linear correlation between the current and the electric field
 - Maxwellian background for the equilibrium of the distribution function
- **Remove**
 - Homogeneous
 - Unbounded
- **Introduce**
 - Cold plasma ($T \longrightarrow 0$)

Fourier transform in time for the perturbed quantities

$$\tilde{\vec{E}}(\omega, \vec{r}) = \int dt \vec{E}(\vec{r}, t) e^{i\omega t}$$

$$\tilde{f}_1(\omega, \vec{r}) = \int dt f_1(\vec{r}, t) e^{i\omega t}$$

- The wave equation reduces to the vector-(quasi)Helmholtz equation

$$\nabla \wedge \nabla \wedge \tilde{\vec{E}}(\vec{r}, \omega) + \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}}^H(\vec{r}, \omega) \cdot \tilde{\vec{E}}(\vec{r}, \omega) = \cancel{-i \frac{\omega^2}{c^2} \underline{\underline{\varepsilon}}^A(\vec{r}, \omega) \cdot \tilde{\vec{E}}(\vec{r}, \omega)}$$

$$\vec{E}(\vec{r}, t) = \int d\omega \vec{E}(\vec{r}) \delta(\omega - \omega_0) e^{-i\omega t} = \vec{E}(\vec{r}) e^{-i\omega_0 t}$$

$$\nabla \wedge \nabla \wedge \tilde{\vec{E}}(\vec{r}, \omega) + \frac{\omega^2}{c^2} \tilde{\underline{\underline{\varepsilon}}}^H(\vec{r}, \omega) \cdot \tilde{\vec{E}}(\vec{r}, \omega) = 0$$

- The solution of the equation by asymptotic techniques (WKB expansion) simplifies the treatment of the problem but can only be applied when the wavelengths are much smaller than the typical scale of macroscopic parameter variation (density, magnetic field, etc.)
- **This simplification allows, however, to have a more accurate description of the geometry and configuration of the plasma where the waves are propagating.**
- **The presence of singularities (cut-offs, mode conversions, reflection layers, etc.) represent a limitation of the approach**

- **WKB expansion of the field**

$$\vec{E}(\vec{r}) \sim \vec{A}(\vec{r}) \exp[ih^{-1}S(\vec{r})]$$

- **A is the amplitude and S is the phase**
- **A e S are functions that vary on different spatial scales. The phase varies on a much smaller scale than the amplitude; the expansion parameter h represents the ratio between the wavelength and the spatial size of the plasma.**

- Different equations can be written at different orders in development in $\hbar$
- At the lowest order $O(\hbar^{-2})$ it is possible to write an equation for the S phase function

$$-\vec{A}(\vec{r})[\nabla S(\vec{r})]^2 + \vec{A}(\vec{r}) \cdot \nabla S(\vec{r}) \nabla S(\vec{r}) + \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}^H(\vec{r}) \cdot \vec{A}(\vec{r}) = 0$$

- At the next order ($\hbar^{-1}$) it is possible to write an equation for the transport of amplitude

$$\vec{A} \cdot \nabla \nabla S(\vec{r}) - \vec{A} \nabla \cdot \nabla S(\vec{r}) + \nabla S(\vec{r}) \nabla \cdot \vec{A} - 2 \nabla S(\vec{r}) \cdot \nabla \vec{A} + \nabla \vec{A} \cdot \nabla S(\vec{r}) = \frac{\omega^2}{c^2} \underline{\underline{\epsilon}}^A(\vec{r}) \cdot \vec{A}(\vec{r})$$

- [to treat this equation see for example: *R. Spigler, M. Vianello, Liouville–Green Asymptotics for almost-diagonal second-order matrix differential equations, Asymptotic Analysis 48 267 (2006)*]

Equation for the phase: Hamilton-Jacobi equation

$$H \equiv \det \left[-(\nabla S(\vec{r}))^2 \underline{\underline{I}} + \nabla S(\vec{r}) \nabla S(\vec{r}) + \underline{\underline{\varepsilon}}^H(\vec{r}) \right] = 0$$

H is the Hamiltonian

S is the Hamilton's principal function

The Hamilton's equations are the characteristics of Hamilton-Jacobi's equation

$$\vec{p} = \nabla S$$

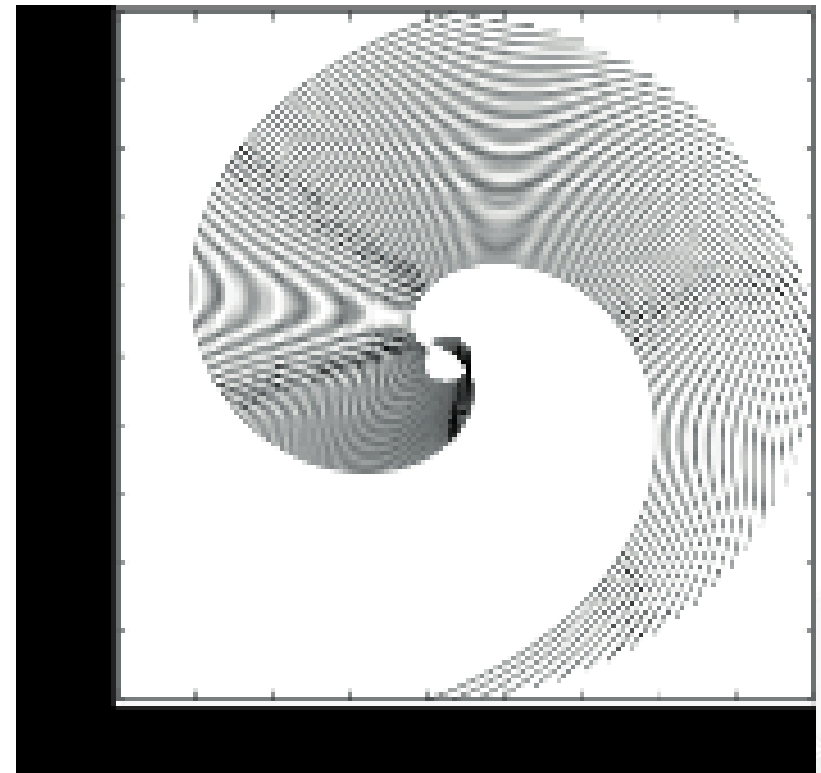
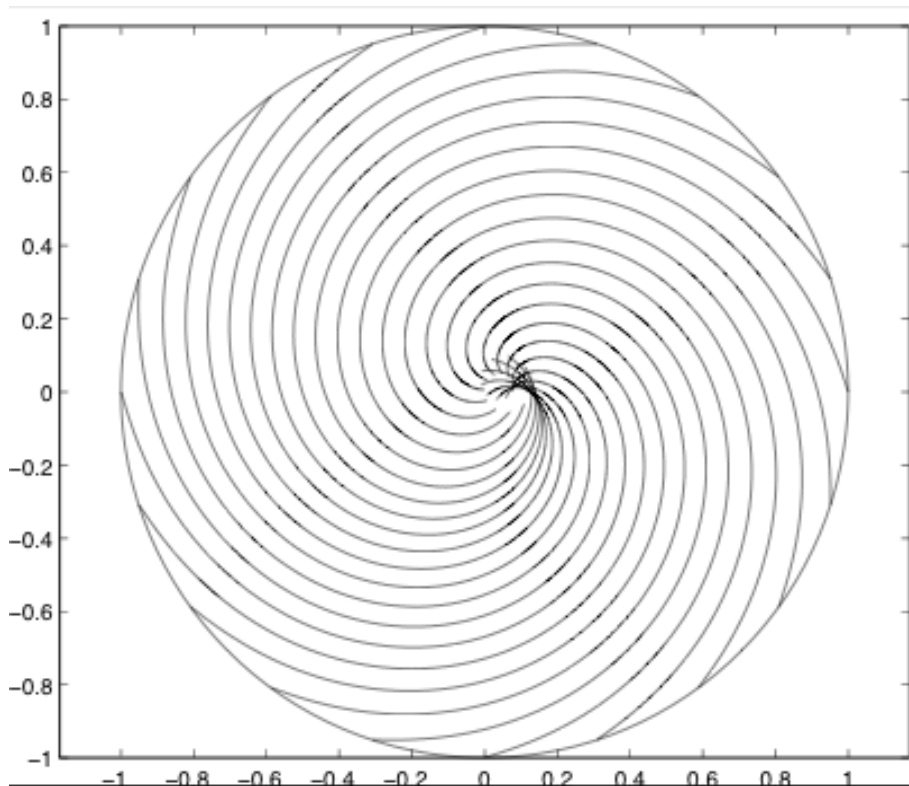
$$\dot{\vec{r}} = \partial_{\vec{p}} H(\vec{r}, \vec{p}, \omega)$$

$$\dot{\vec{p}} = -\partial_{\vec{r}} H(\vec{r}, \vec{p}, \omega)$$

$$dS = \vec{p} \cdot d\vec{r}$$

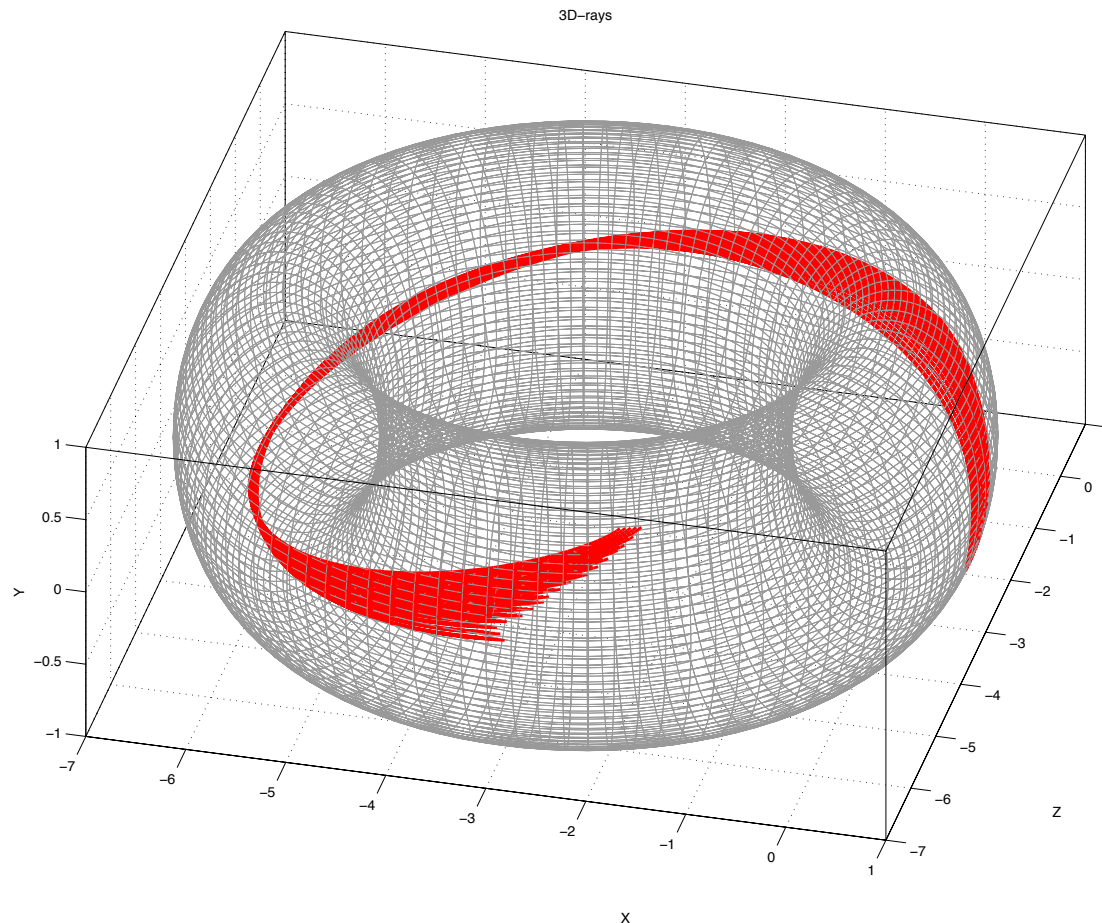
Solution of Hamilton equations for the Lower Hybrid Wave propagation (phase reconstruction through the rays)

"Ray tracing" in a poloidal cut of the torus (FTU plasma case)

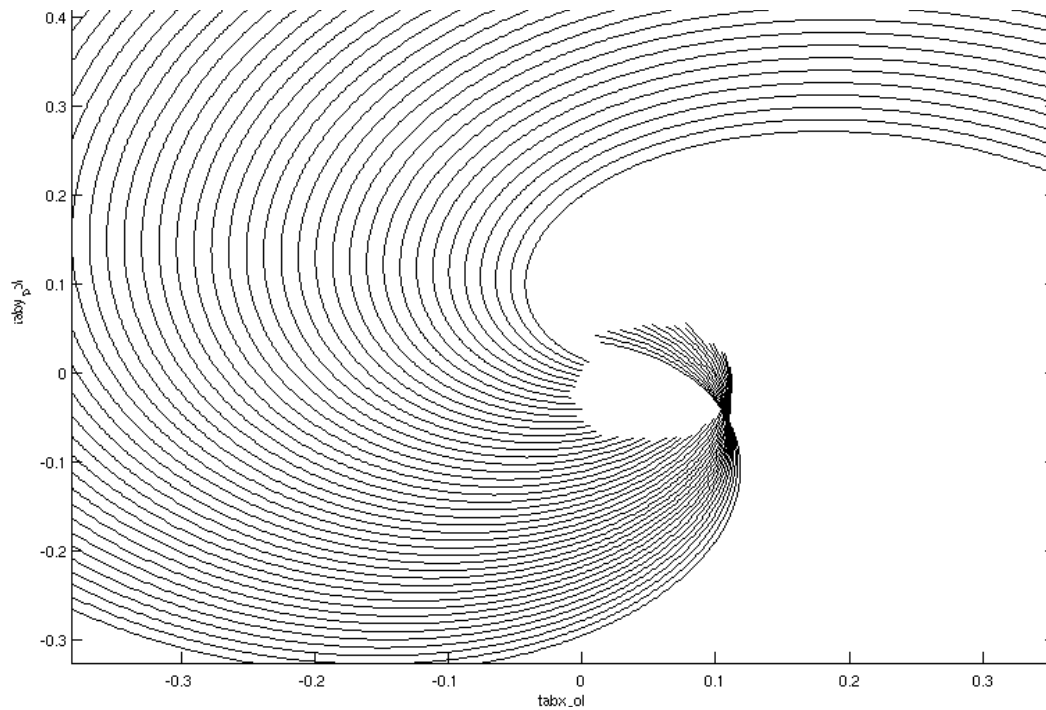


Solution of Hamilton equations for the Lower Hybrid Wave propagation (phase reconstruction through the rays)

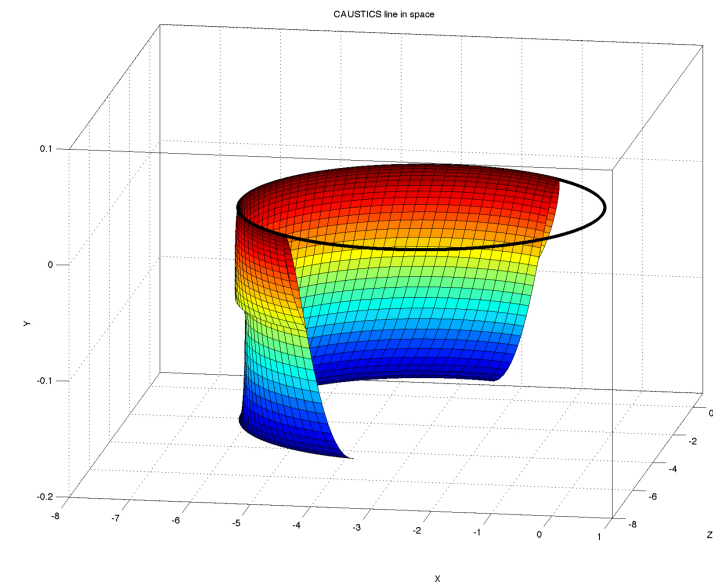
"Ray tracing" of a group of rays starting from the antenna located at the edge of the plasma (3D plot)



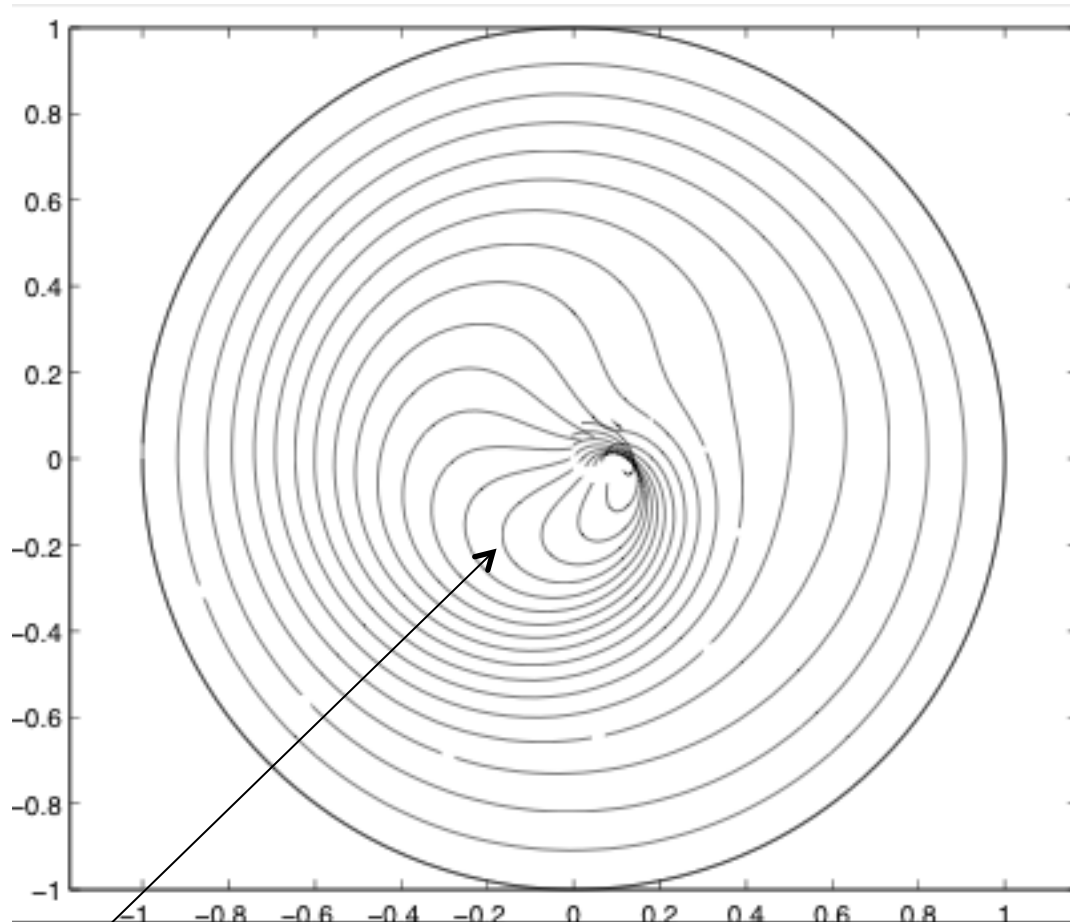
Formation of a caustic near the center of the plasma for the LHW propagation



Caustic is the envelope of the rays reflecting in a plasma area. It is a surface in space so that all the rays are tangent to it

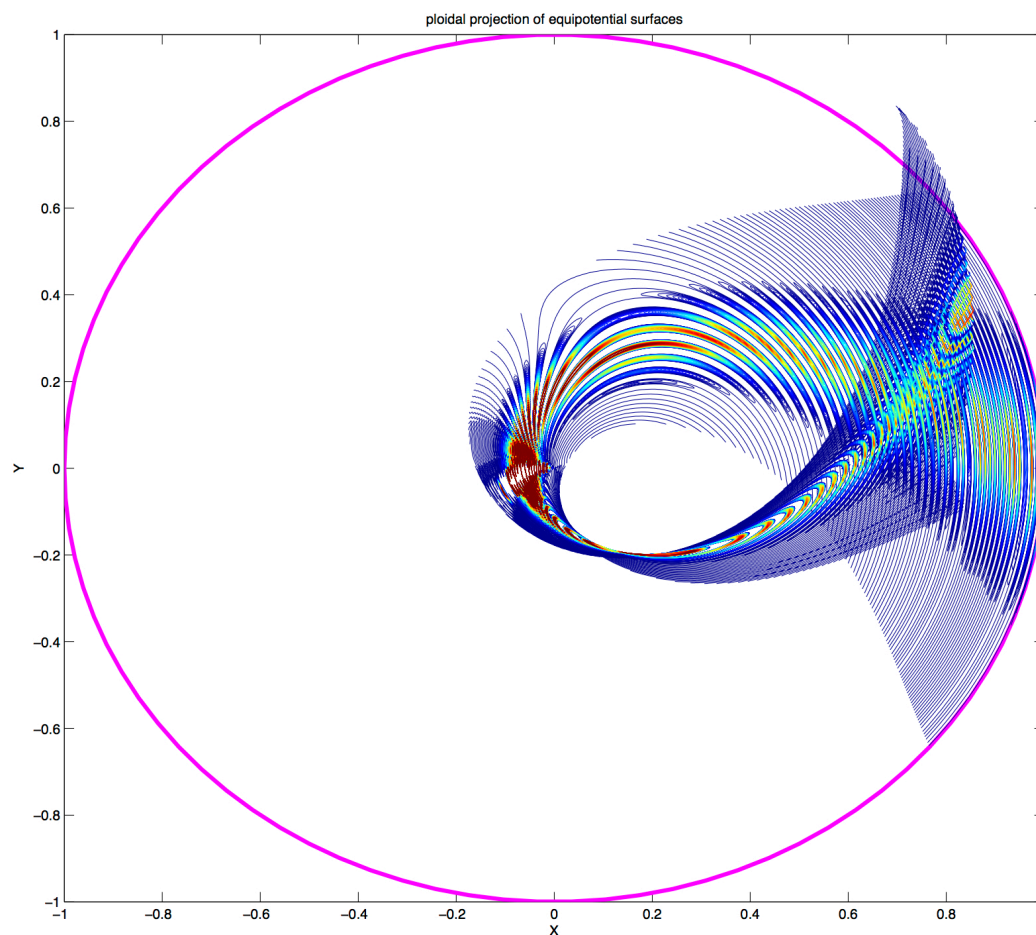
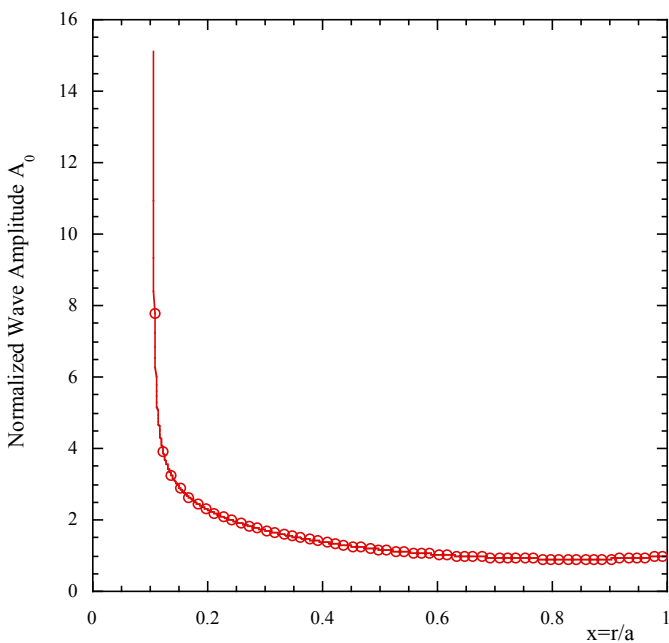


Evolution of the wave-front from the antenna to the center of the plasma



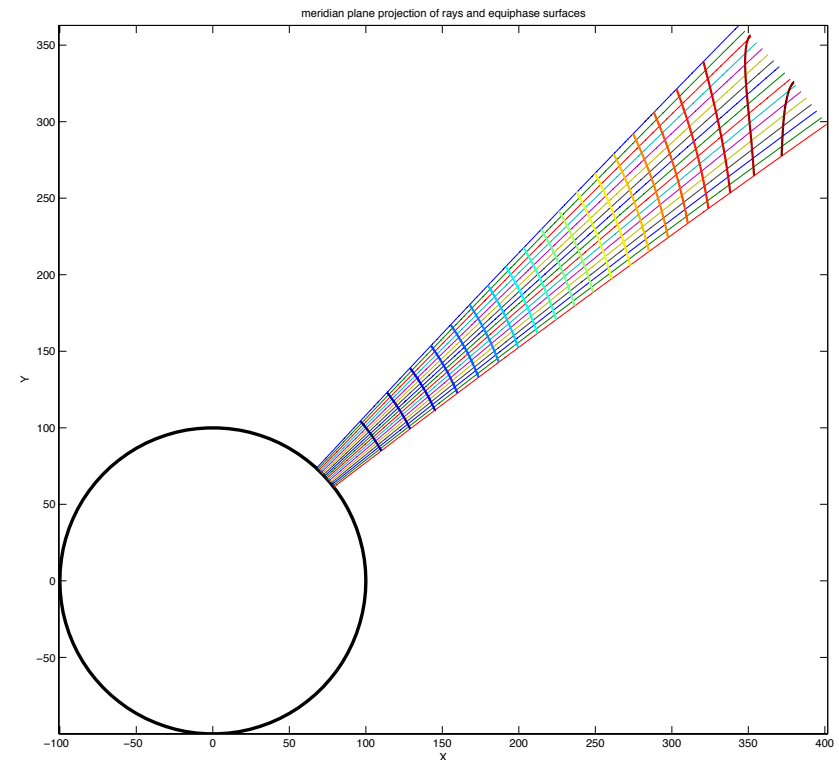
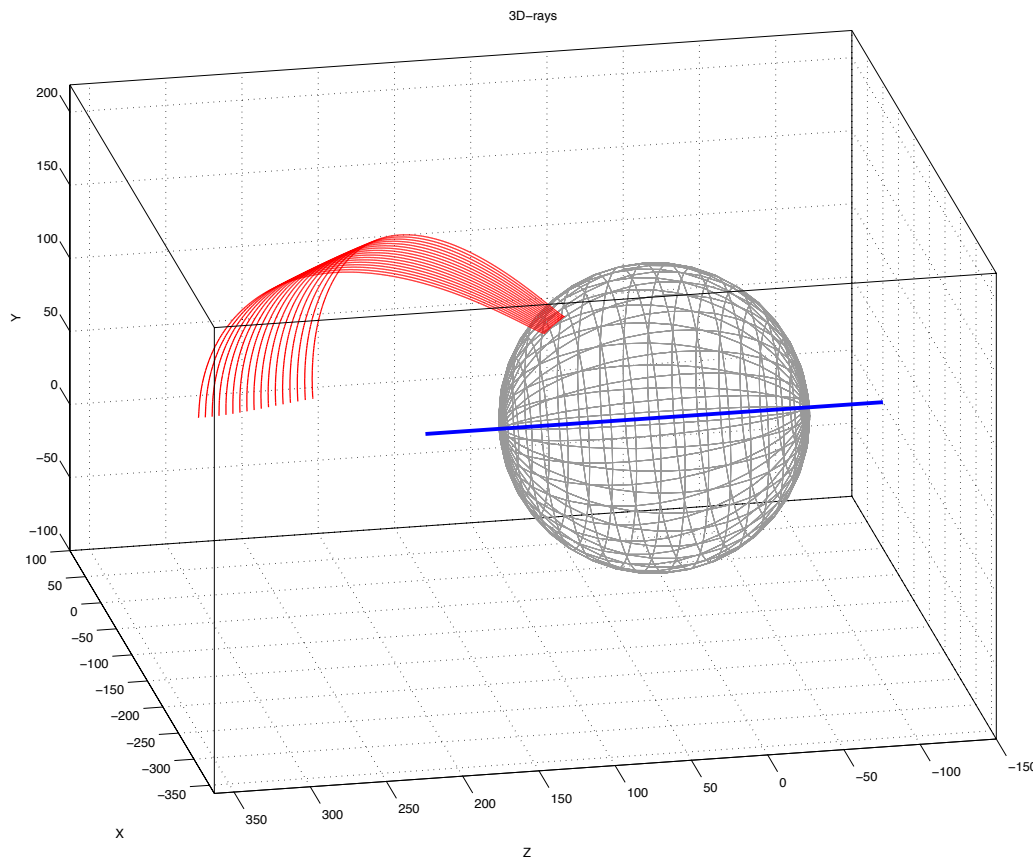
Distortion of the surface due to toroidicity

Evolution of the amplitude and its divergence on the caustic in 1 and 2 Dimensions



Radio signal propagation in the ionosphere

As an example of propagation of a radio signal in the ionosphere we simulated a signal emitted by the Omega-Navigation Transmitter station (10kW) in North Dakota (*E. Camporeale, GL Delzanno, P. Colestock, Journal of Geophysical Research: Space Physics 117 A10 2012*) to explain the loss of signal received by the ISEE1 satellite



Conclusions



- The physics of generation, propagation & absorption of an electromagnetic wave in a plasma (possibly magnetically confined) is modelled by the equation system Maxwell-Vlasov (kinetic approach for the plasma (complete))
- This model presents difficulties to be solved owing to
 - of the integer-differential nature of the system
 - Geometry
 - Inhomogeneous, non-stationary and plasma boundary
 - Non-linearity
- Useful information can be extracted by removing the above features and working with Fourier analysis
- Dispersion relation
 - "complete dispersion relation" kinetic case (temperature effects)
 - "cold dispersion relation" (without temperature effects, cold limit)

Conclusions



- Attempt to solve the Wave Equation:
- The linear and cold limit of the Maxwell-Vlasov system leads to a system of differential equations for partial-type derivatives of the hyperbolic type for the electromagnetic field.
- The solution is given by an asymptotic WKB expansion of the electric field.
- At the lowest order in the expansion parameter we get a partial differential equation for the phase function S , formally analogous to that of Hamilton-Jacobi in mechanics.
- At the next order we get an equation for the transport of the amplitude
- Many properties of propagation can be deduced in a realistic geometry without having to solve the wave equations directly.



Soluzioni innovative per il calcolo numerico di campi elettromagnetici in configurazioni d'interesse per la fusione nucleare

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Outline

Background & motivation

**Modeling electro-magneto-
mechanical transients**

**Nonlinear evolution of fusion
plasmas**

Conclusions

Background & motivation /1

- Plasma evolution described by the MHD equations
- ***Plasma Unstable Modes***, time constants order ***microseconds***
- Stabilizing effects due to the induced currents in the passive conducting structures
- because of finite conductivity, induced currents decay $\square$ modes still unstable, time constants order ***milliseconds***: Vertical displacement events (VDE) , Resistive Wall Modes (RWM)
- **active control** of plasma position and shape
- accurate numerical models of the 3D conducting structures $\square$ **very high CPU time and memory needs**

Background & motivation /2

Future magnetic confinement fusion devices have high performances □ special care in design

Disruptions are a concern: sudden loss of magnetic confinement cause significant electromagnetic loads on conductors

Electro-magneto-mechanical transients can be an important issue

Several plasma modelling approaches available

2D linearized (CREATE_L), 2D nonlinear (DINA, TSC, MAXFEA), 3D nonlinear (M3D, JOREK) & many others

Modelling of **conductors: usually simplified** (2D, thin wall etc.)

The **plasma evolution** may be affected by oversimplifications

The **EM load** evaluation requires a second computation on a detailed 3D mesh of conductors (not self-consistent)

Background & motivation /3

Computational tool	Plasma description	Structures
PROTEUS, MAXFEA, CREATE_NL	Axisymmetric, Evolutionary equilibrium	Axisymmetric (Finite elements)
DINA	Axisymmetric, Evolutionary equilibrium	Axisymmetric (Filaments)
TSC	Axisymmetric, MHD equations	Axisymmetric (Finite Differences)
CREATE_L, PET	Axisymmetric, Linearized perturbed equilibrium	Axisymmetric (Finite elements)
MARS, MARS-F	Three-dimensional, Linearized MHD	Axisymmetric (thin wall)
VALEN, STARWALL	Three-dimensional, Linearized MHD	Three dimensional (thin wall)
CarMa	Three-dimensional, Linearized MHD	Three dimensional (volumetric)
M3D, JOREK	Three-dimensional, Nonlinear MHD	Axisymmetric (thin wall)
ANSYS (CHEN)	Axisymmetric Filaments with imposed currents	Three dimensional (volumetric) + dynamics
CARIDDI	Axisymmetric Filaments and 3D conductors with imposed currents, Imposed halo currents	Three dimensional (volumetric) + dynamics
CarMa0NL	Axisymmetric, Evolutionary equilibrium	Three dimensional (volumetric)

Outline

Background & motivation

Modeling ~~electro~~-magneto-
mechanical transients

Nonlinear evolution of fusion
plasmas

Conclusions

Complexity of 3D passive structures

- Need of highly-accurate computational models for the simulation of the interaction between plasma and **3D** passive conducting structures
 - to analyse the plasma evolution
 - to design the plasma position and shape control
 - to evaluate the electromagnetic loads
- Limits of the « commercial » codes
 - very good for the mechanical analysis of the structures
 - reasonable performances in the conventional electromagnetic analysis, with the usual necessity to discretize the free space

Outline

Background & motivation

**Modeling electro-magneto-mechanical
transients**

Magneto-quasi-static Integral
formulation of the field equations

Vibration of a conducting structure
in the presence of high magnetic
field

ITER vacuum vessel under a slow
AVDE

Nonlinear evolution of fusion plasmas

Conclusions

Magneto-quasi-static Integral formulation /1

$$\left(\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \Rightarrow \right) \quad \int_0^t \mathbf{E}(\mathbf{r}, \tau) d\tau = - \mathbf{A}(\mathbf{r}, t) - \mathbf{A}_s(\mathbf{r}, t) - \nabla U$$

$$(\nabla \times \mathbf{H} = \mathbf{J} \Rightarrow) \quad \text{with } \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V_c} \frac{\mathbf{J}(\mathbf{r}', t)}{|\Delta \mathbf{r}(t)|} dV' = \mathbf{A}[\mathbf{J}]$$

$$\int_0^t \int_{V_c} \mathbf{W} \cdot (\mathbf{E} - \sigma^{-1} \mathbf{J}) d\tau = 0 \quad \mathbf{J} \in \mathbf{H}_0(V_c, \text{div}) \quad \forall \mathbf{W} \in \mathbf{H}_0(V_c, \text{div})$$

$$\mathbf{H}_0(D, \text{div}) \triangleq \left\{ \mathbf{J} \in \mathbf{H}(D, \text{div}), \nabla \cdot \mathbf{J} = 0 \text{ in } D, \mathbf{J} \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial D \right\}$$

$$\mathbf{H}(D, \text{div}) \triangleq \left\{ \mathbf{u} \in \mathbf{L}^2(D) : \nabla \cdot \mathbf{u} \in \mathbf{L}^2 \right\}$$

$$\frac{d}{dt} \left[\int_{V_c} \mathbf{W} \cdot (\mathbf{A}[\mathbf{J}] + \mathbf{A}_s + \nabla U) d\tau \right] + \int_{V_c} \mathbf{W} \cdot \sigma^{-1} \mathbf{J} d\tau = 0,$$

Magneto-quasi-static Integral formulation /2

$$\frac{d}{dt} \int_{V_c} \mathbf{W} \cdot \mathbf{A}[\mathbf{J}] d\tau + \int_{V_c} \mathbf{W} \cdot \sigma^{-1} \mathbf{J} d\tau + \frac{d}{dt} \int_{V_c} \mathbf{W} \cdot \mathbf{A}_s d\tau + \sum_{h=1, N_E} \int_{S_h} \phi_h \mathbf{W} \cdot \hat{\mathbf{n}} dS = 0$$

$$\mathbf{J}(\mathbf{r}, t) = \sum_e I_k(t) \mathbf{J}_k(\mathbf{r}) \quad \mathbf{J}_k \in S, \quad \mathbf{W}_k = \mathbf{J}_k$$

$$\frac{d(LI)}{dt} + RI + H^T \Phi + \frac{dE^s}{dt} = 0$$

$$L_{i,j} \cong \frac{\mu_0}{4\pi} \int_{V_c} \int_{V_c} \frac{\mathbf{W}_i(\mathbf{r}) \cdot \mathbf{W}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r} d\mathbf{r}'$$

$$H_{k,i} = \int_{S_k} \mathbf{W}_i \cdot \hat{\mathbf{n}}_k dS \quad R_{i,j} \cong \int_{V_c} \mathbf{W}_i(\mathbf{r}) \cdot \sigma^{-1} \cdot \mathbf{W}_j(\mathbf{r}) d\mathbf{r}$$

Magneto-quasi-static Integral formulation /3

$$L \frac{dI}{dt} + RI + H^T \Phi + \frac{dE_s}{dt} = 0$$

$$HI = I_h$$

- 1) Initialization $O(N)$
- 2) Computation of L [$O(N^2)$], R , V
- 3) LU decomposition of $L + \text{rot}R$ [$O(N^3)$].
- 4) computation of the transient
- 5) Output $O(N)$

TERM	Assembly Time	Memory Required	Computation Time
R	LOW	LOW	LOW
L	HIGH	HIGH	HIGH

Scalability

$$\mathbf{Z} \mathbf{I} = \mathbf{U}$$

Z is a full $N \times N$ matrix **memory: $O(N^2)$**

direct inversion: $O(N^3)$

Iterative Inversion **needs $\mathbf{Z}\mathbf{I}$ products: $O(N^2)$**
(with preconditioning)

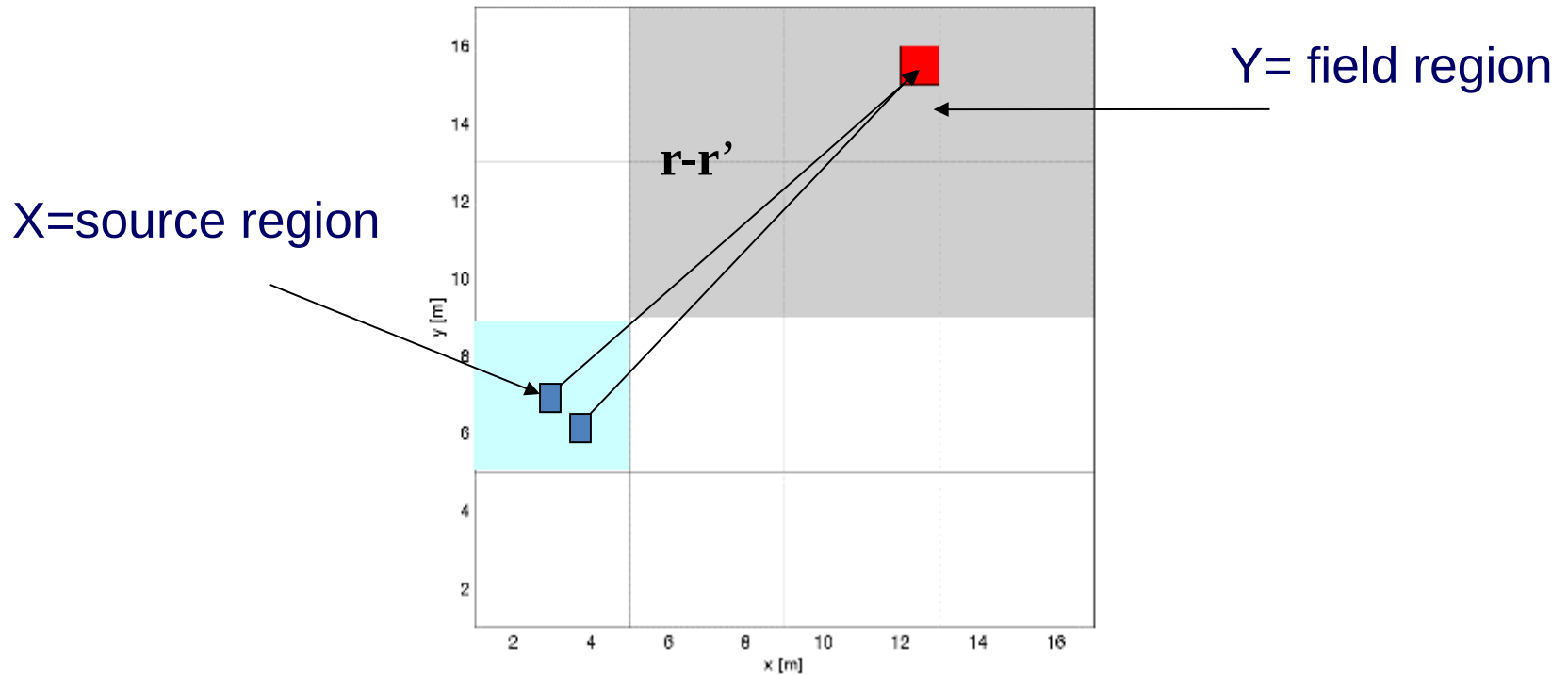
CPU time for serial computation $T_s(N) = O(N^2)$

With p processors the ideal CPU time is $T_p(N) = O(N^2/p)$

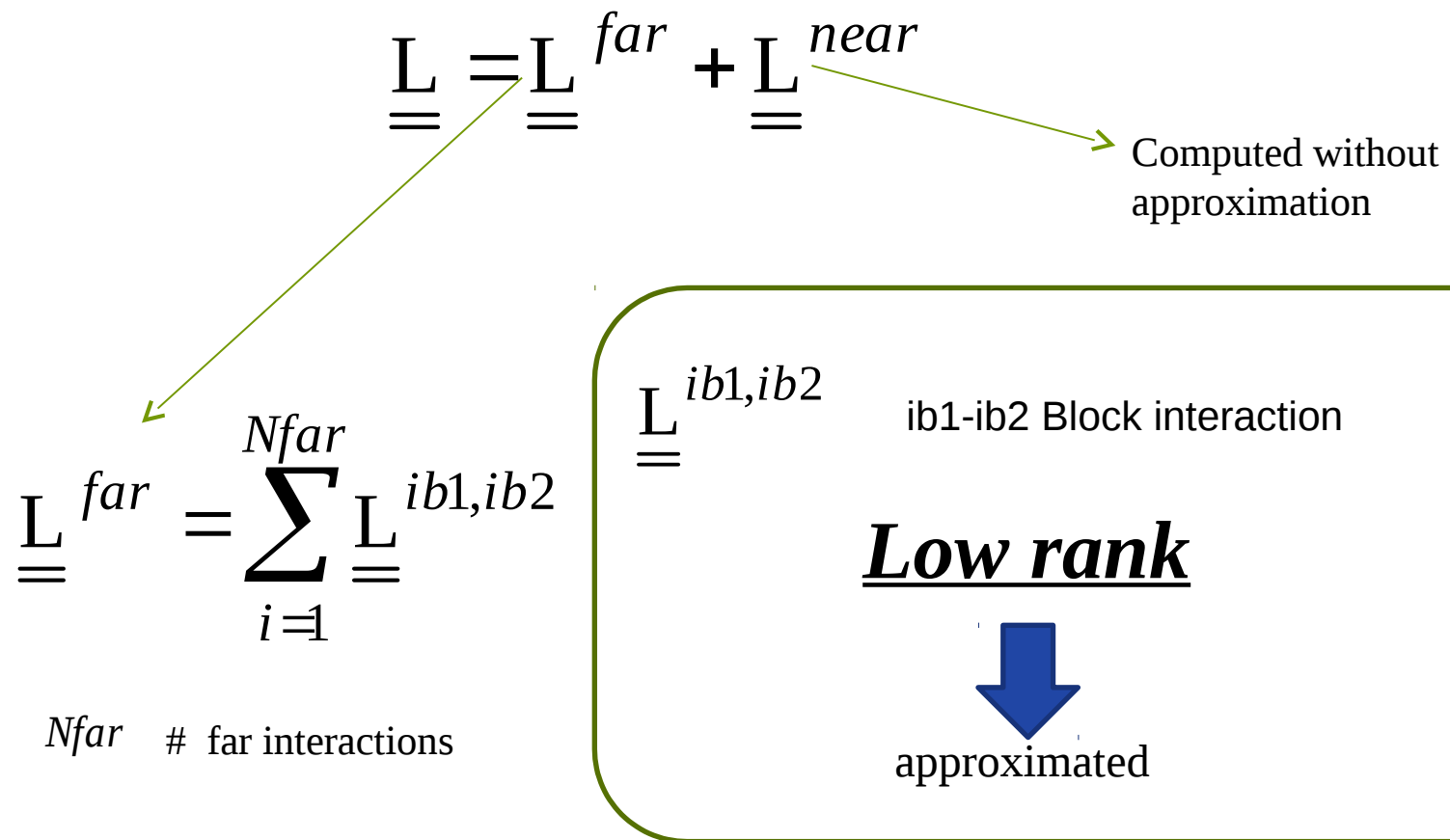
$$T_s(N_s) = T_p(N_p) \quad \square \quad N_p = N_s \sqrt{p}$$

Recursive block SVD method /1

$$L_{ij}^{XY} = \frac{\mu_0}{4\pi} \int_Y \int_X \frac{\mathbf{W}_i(\mathbf{r}) \cdot \mathbf{W}_j(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv dv', \quad \begin{array}{l} \mathbf{W}_i \neq 0 \text{ in } Y \\ \mathbf{W}_j \neq 0 \text{ in } X \end{array}$$



Recursive block SVD method /2



Recursive block SVD method /3

$\underline{\underline{\mathbf{L}}}^{XY}$ and $\underline{\underline{\mathbf{D}}}^{XY}$ are low rank matrices.

Rank r decreases as the separation between X and Y increases

$$\mathbf{L}^{XY} = \mathbf{Q}^X \mathbf{R}^Y$$

$$\dim(\mathbf{L}^{XY}) = m \times n$$

$$\dim(\mathbf{Q}^X) = m \times r$$

$$\dim(\mathbf{R}^Y) = r \times n$$

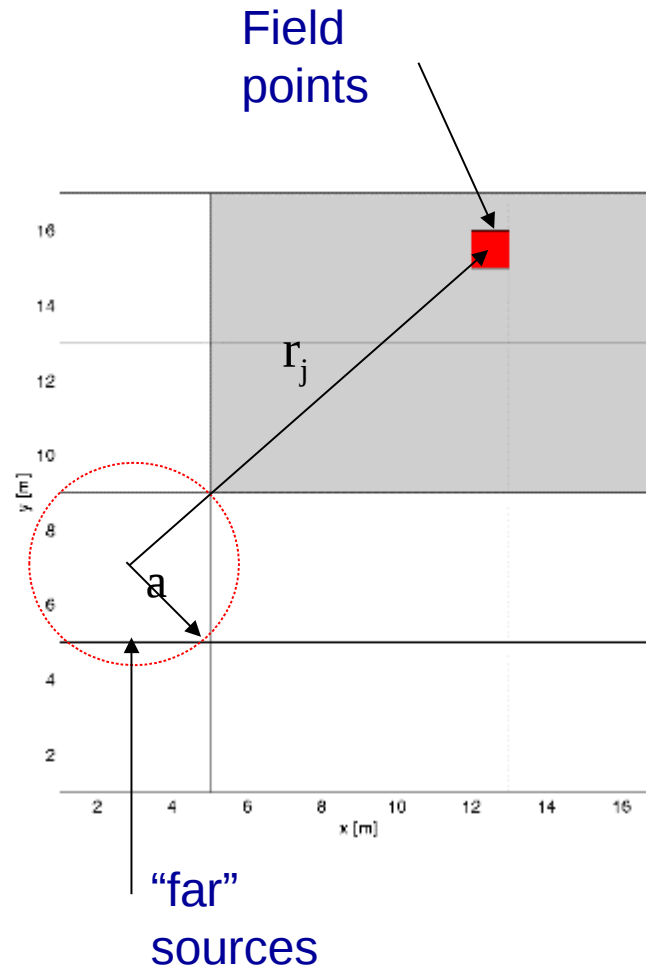
$$\mathbf{L}^{XY} \mathbf{I}^Y \Rightarrow m \times n \text{ operations}$$

$$\mathbf{Q}^X \mathbf{R}^Y \mathbf{I}^Y \Rightarrow r \times (n + m) \text{ operations}$$

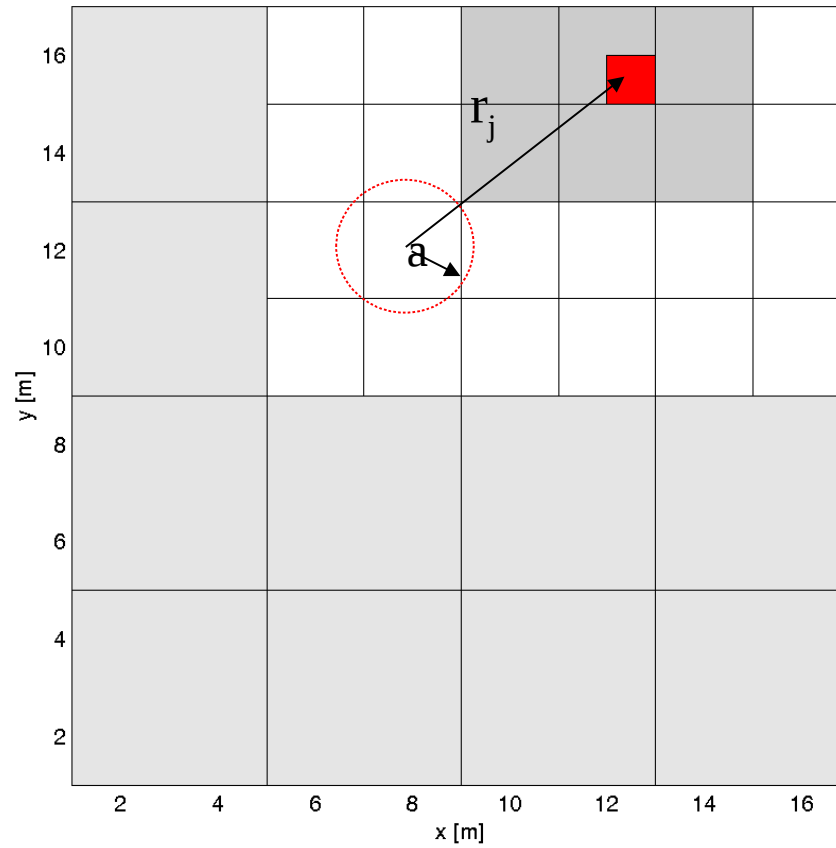
$$r \ll m, n$$

$$\text{Memory Required and ComputationTime} = (m + n) \times r$$

Recursive block SVD method /4

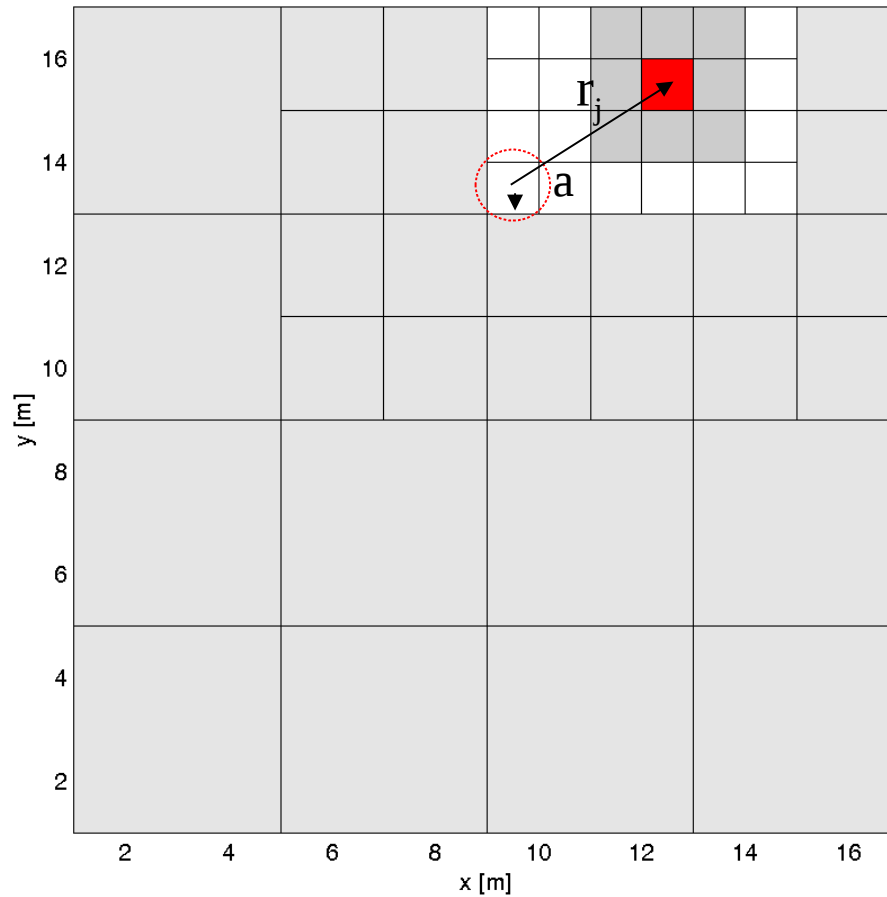


Recursive block SVD method /5

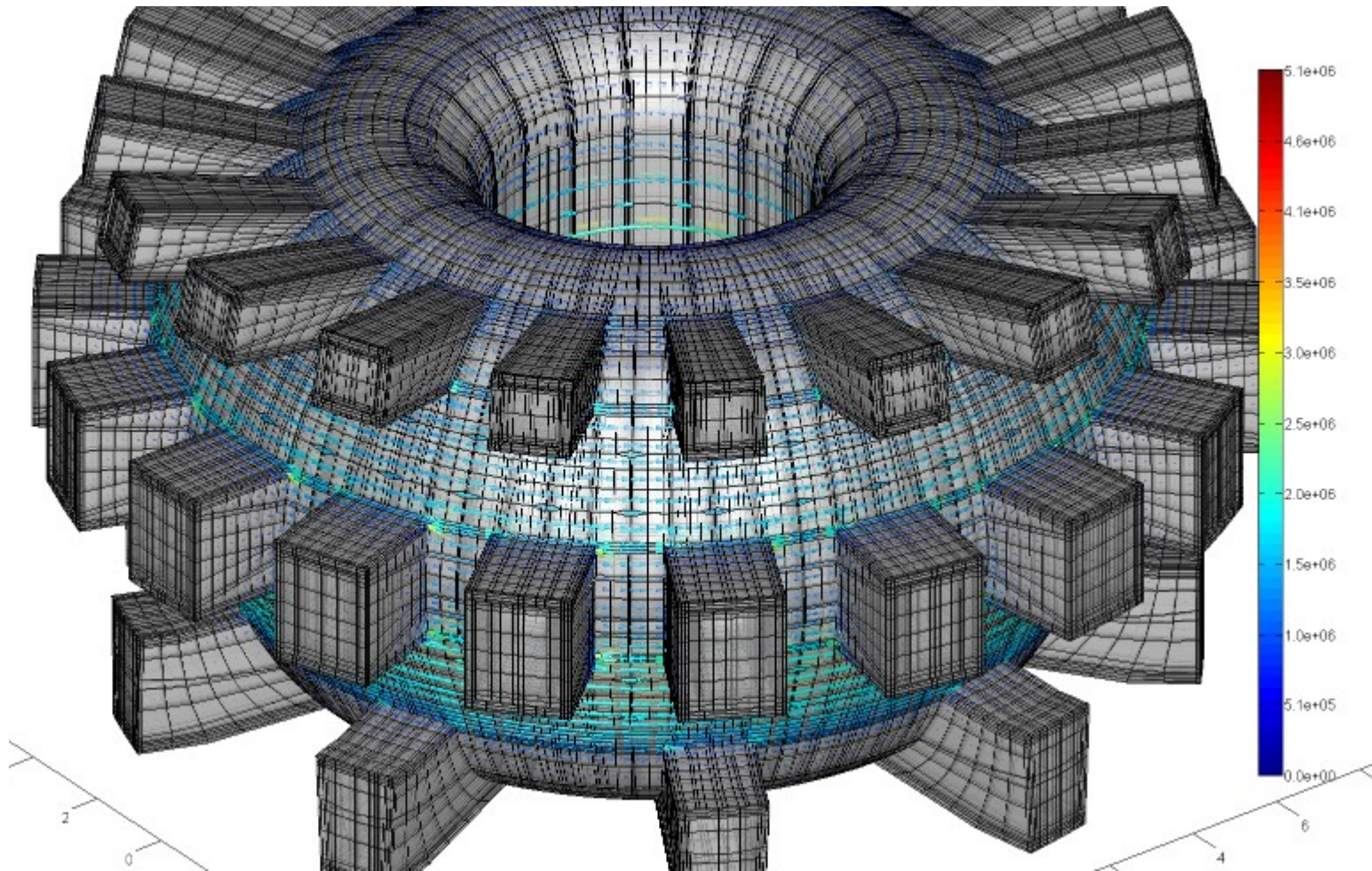


Recursive block SVD method /6

The product ZI scales as $N \log N$



ITER VV under a slow VDE cat III



Nodes: 323.676, elements: 172.980; DOFs: 187.885, 100
processors, CPU time: 5h

n-fold rotational symmetry/1

Let S denote the $2\pi/n$ rotation around the axis of symmetry, such as $S^k x$ is the rotated image of a point x .

$$I = \{1, \dots, N\} = \{1, \dots, Nc, \sigma(1), \dots, \sigma(Nc), \dots, \sigma^{n-1}(1), \dots, \sigma^{n-1}(Nc)\}$$

$$Z_{I,I} = L_{I,I} + \Delta t R_{I,I}$$

$$\begin{bmatrix} Z_{I,I} & Z_{I,Iadd} \\ Z_{Iadd,I} & Z_{Iadd,Iadd} \end{bmatrix} \begin{bmatrix} I_I^m \\ I_{Iadd}^m \end{bmatrix} = \begin{bmatrix} L_{I,I} & L_{I,Iadd} \\ L_{Iadd,I} & L_{Iadd,Iadd} \end{bmatrix} \begin{bmatrix} I_I^{m-1} \\ I_{Iadd}^{m-1} \end{bmatrix} + \begin{bmatrix} V_{0_I}^m \\ V_{0_{Iadd}}^m \end{bmatrix}$$

Matrix $Z_{I,I}$ have the following block-circulant form:

$$Z_{I,I} = \begin{pmatrix} Z_1 & Z_2 & Z_3 & \dots & Z_n \\ Z_n & Z_1 & Z_2 & \dots & Z_{n-1} \\ Z_{n-1} & Z_n & Z_1 & \dots & Z_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ Z_2 & Z_3 & Z_4 & \dots & Z_1 \end{pmatrix}$$

n-fold rotational symmetry/2

$$\det(P - \lambda I) = 1 - \lambda^n = 0 \rightarrow \lambda = \sqrt[n]{1} = \omega^k, \omega = e^{j\frac{2\pi}{n}}, k = 0:n-1$$

$$P = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$v_k = [1, \omega^k, \omega^{2k}, \dots, \omega^{(n-1)k}]^T$$

$$F_n = \frac{1}{\sqrt{n}} [v_0, v_1, \dots, v_{n-1}]$$

$$\Lambda = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \dots & & \\ & & & \omega^{n-1} & \end{bmatrix}$$

$$PF_n = F_n \Lambda$$

$$P \otimes Z_k = \begin{pmatrix} 0 & Z_k & 0 & \dots & 0 \\ 0 & 0 & Z_k & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & Z_k \\ Z_k & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$Z_{I,I} = \sum_{k=1}^n P^{k-1} \otimes Z_k$$

n-fold rotational symmetry/3

$$(F_n \otimes U_N) \tilde{z}_{I,I} (F_n^* \otimes U_N) I_I^m = b_I^m$$

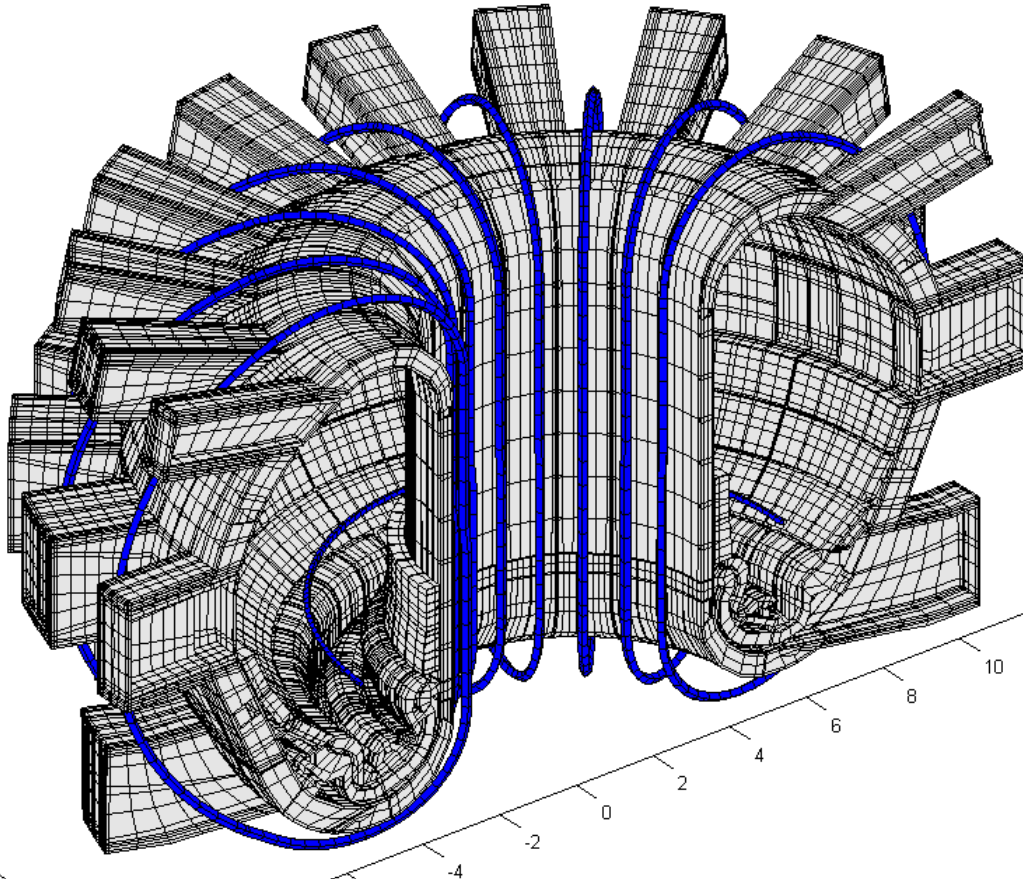
$$\tilde{z}_{I,I} = \sum_{k=1}^n \Lambda^{k-1} \otimes Z_k = \begin{bmatrix} \tilde{z}_1 & 0 & \dots & 0 \\ 0 & \tilde{z}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{z}_n \end{bmatrix}, \tilde{z}_i = \sum_{k=1}^n \omega^{(i-1)(k-1)} Z_k$$

$$\tilde{z}_{I,I} \tilde{i}_I^m = \begin{bmatrix} \tilde{z}_1 & 0 & \dots & 0 \\ 0 & \tilde{z}_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \tilde{z}_n \end{bmatrix} \begin{bmatrix} \tilde{i}_1 \\ \tilde{i}_2 \\ \dots \\ \tilde{i}_n \end{bmatrix} = \tilde{b}_I^m = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \dots \\ \tilde{b}_n \end{bmatrix}$$

$$\tilde{i}_I^m = (F_n^* \otimes U_N) I_I^m, \tilde{b}_I^m = (F_n^* \otimes U_N) b_I^m$$

CPU time and memory needs scale as N^2/n instead of N^2 .

n-fold rotational symmetry/3



Nodes: 273258

Elements: 147132

DOF

n-fold symm. DOFs= 157810

Additional DOFs = 879

DOFs: 158689

Cluster 25 processor MPI

SVD

Setup CPU time: 5.8e3 s

Transient CPU time: 5.7e4 s

Memory: 3.3G real*8

USING ROTATIONAL SYMMETRY

Setup CPU time: 2.6e3 s

Transient CPU time: 2.6e3 s

Memory: 2.7G real*8

Outline

Background & motivation

Modeling electro-magneto-mechanical transients

Magneto-quasi-static Integral formulation of the field equations

Vibration of a conducting structure in the presence of high magnetic field

ITER vacuum vessel under a slow AVDE

Nonlinear evolution of fusion plasmas

Conclusions

Vibration of a conducting structure in the presence of high magnetic field

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

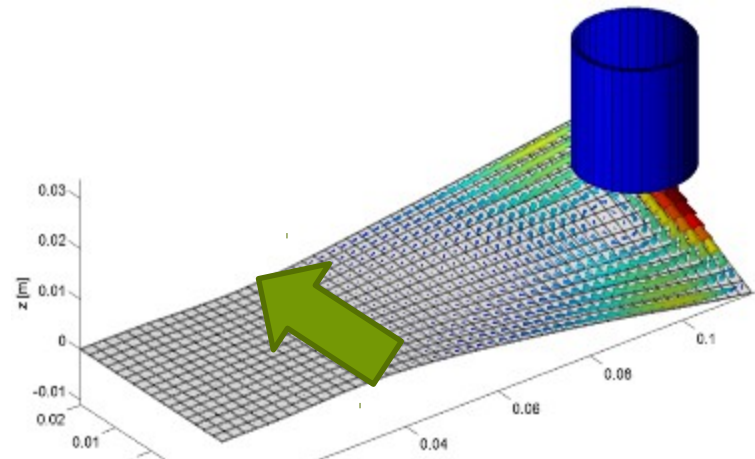
$$L \frac{dI}{dt} + RI - F(t) \frac{du}{dt} = V_0(t)$$



$$\mathbf{f} = \mathbf{J} \times \mathbf{B}$$



$$M \frac{d^2u}{dt^2} + Ku + F^T(t)I = 0$$



Outline

Background & motivation

Modeling electro-magneto-mechanical transients

Magneto-quasi-static Integral formulation of the field equations

Vibration of a conducting structure in the presence of high magnetic field

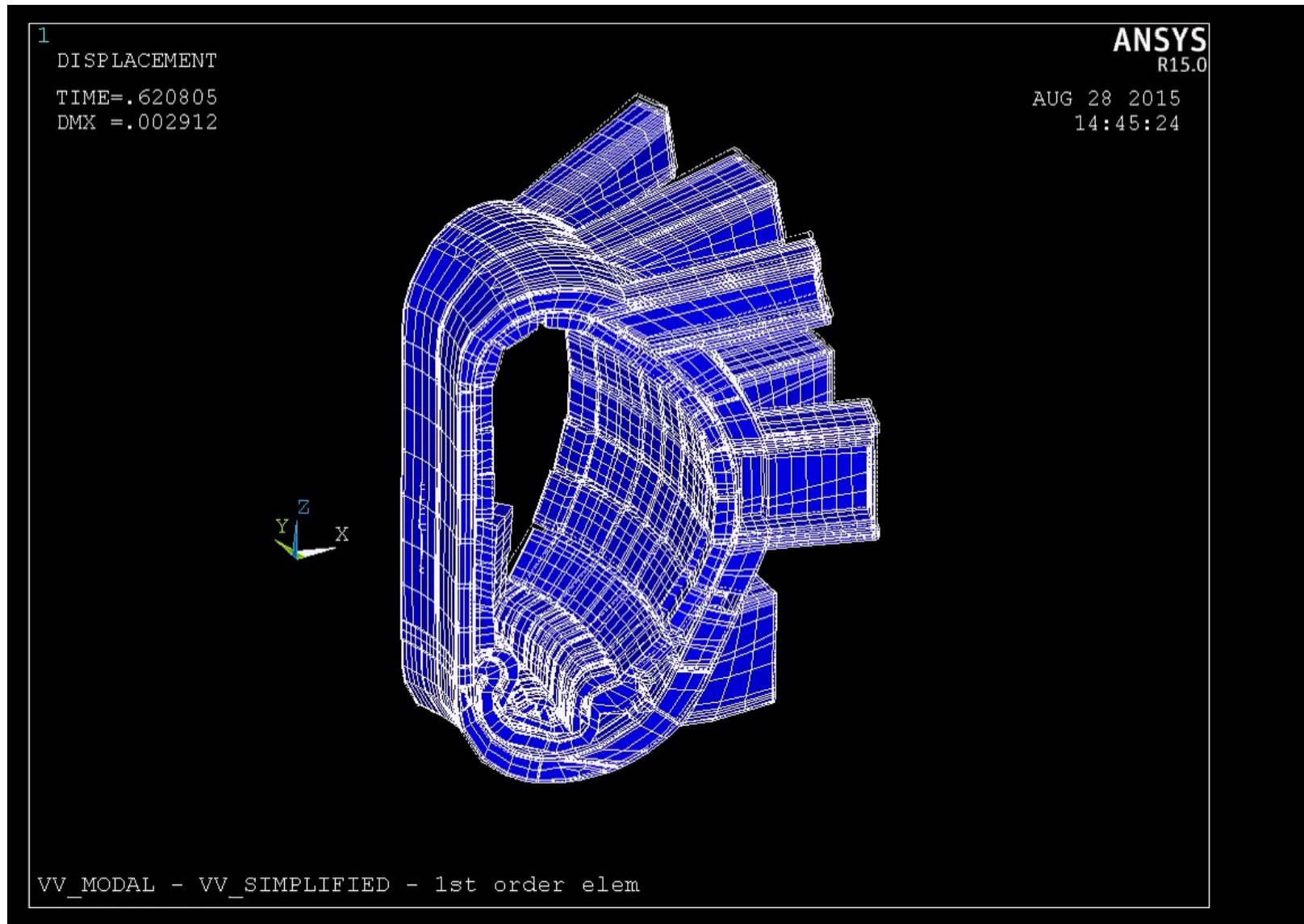


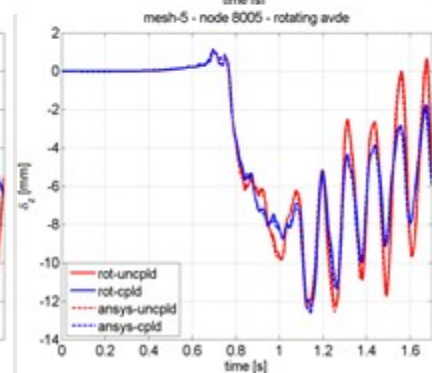
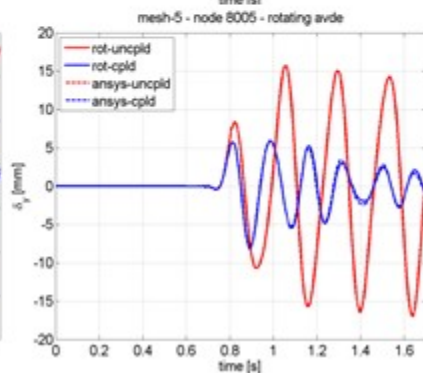
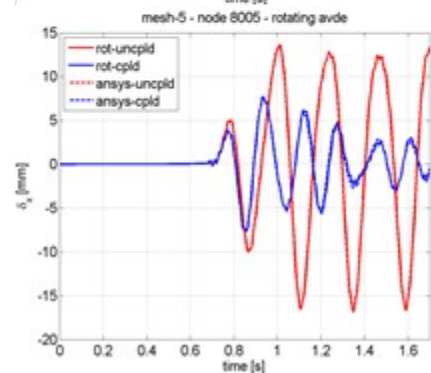
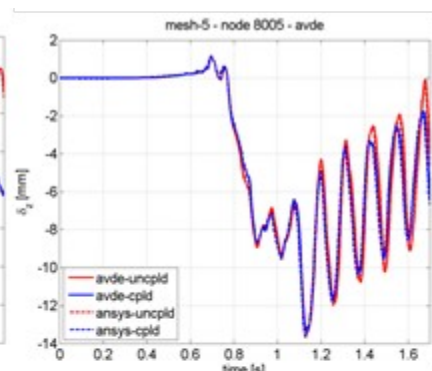
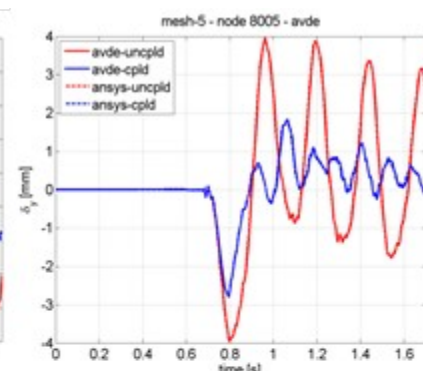
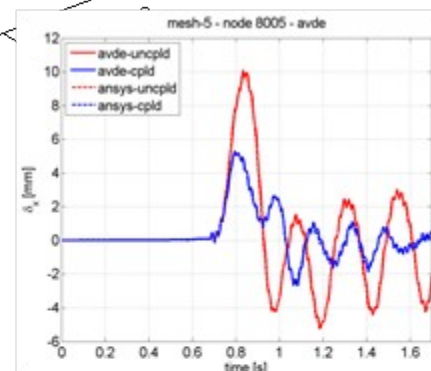
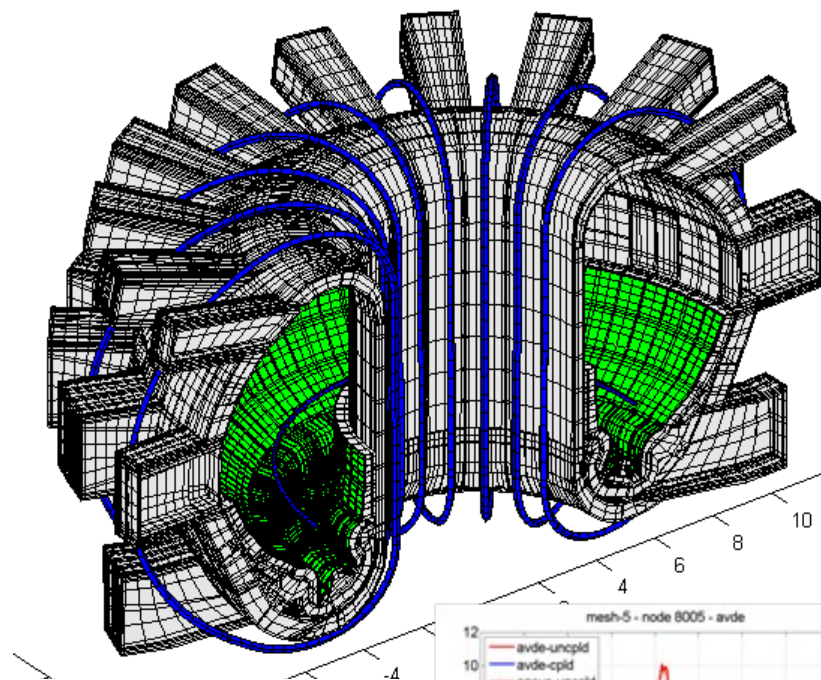
ITER vacuum vessel under a slow AVDE

Nonlinear evolution of fusion plasmas

Conclusions

ITER VV under a slow VDE cat III- results





Outline

Background & motivation

Modeling electro-magneto-mechanical transients

Nonlinear evolution of fusion plasmas

- Mathematical and numerical formulation
- Results and examples of application

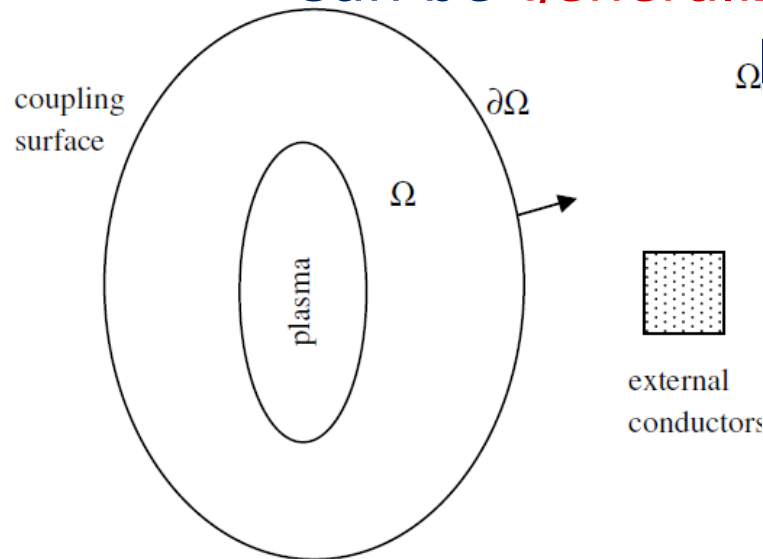
Conclusions

Formulation: the idea

Coupling surface to describe the electromagnetic interaction between the plasma and the conductors.

Different formulations in each domain (the best of both words)

Can be **generalized** to other multiphysics problems



Inside Ω : Grad-Shafranov equations (elliptic nonlinear problem)

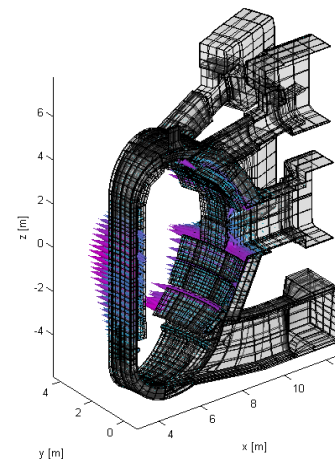
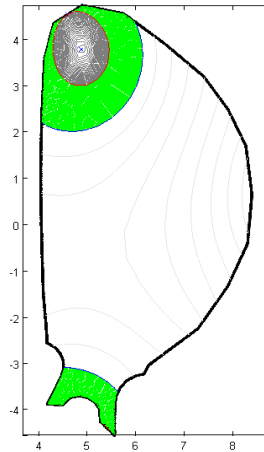
Outside Ω : eddy currents in 3D structures (parabolic linear problem)

On $\partial\Omega$: coupling conditions

Halo currents

Plasma side: solution of Grad-Shafranov equation extended **outside the last closed magnetic surface**, allowing plasma current density on open field lines.

Conductors side: the plasma halo current must be **injected into the structures** through suitable electrical connections.



Outline

Background & motivation

**Modeling electro-magneto-mechanical
transients**

**Nonlinear evolution of fusion
plasmas**

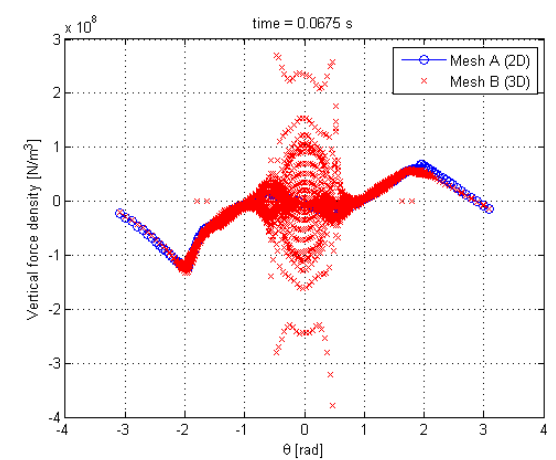
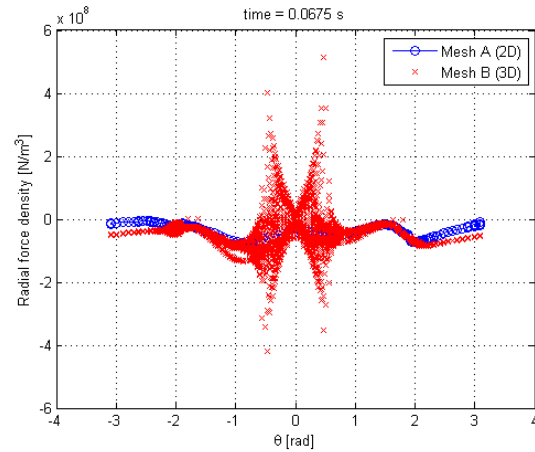
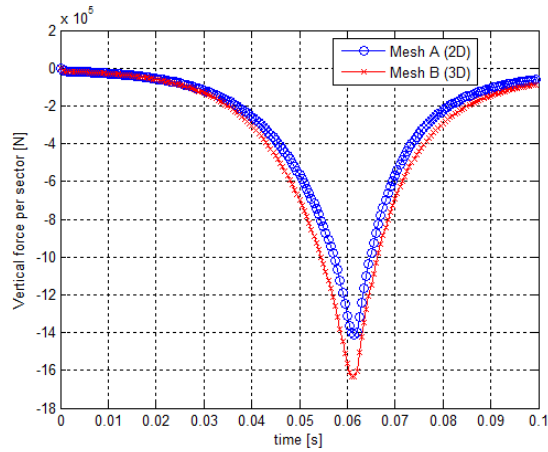
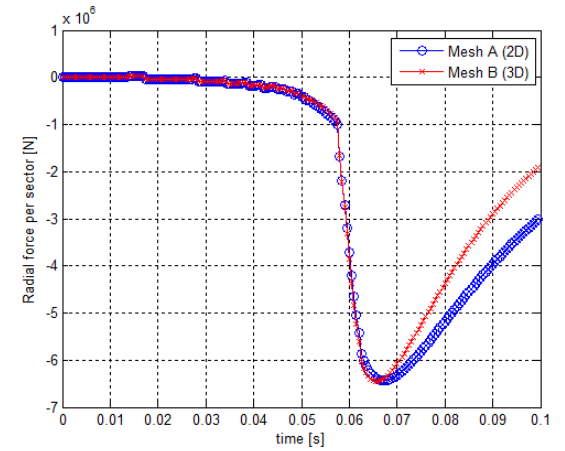
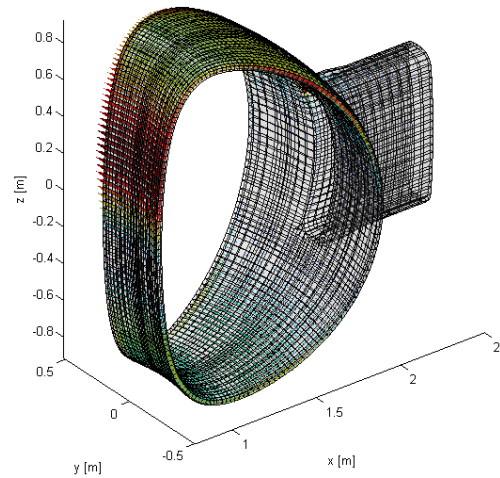
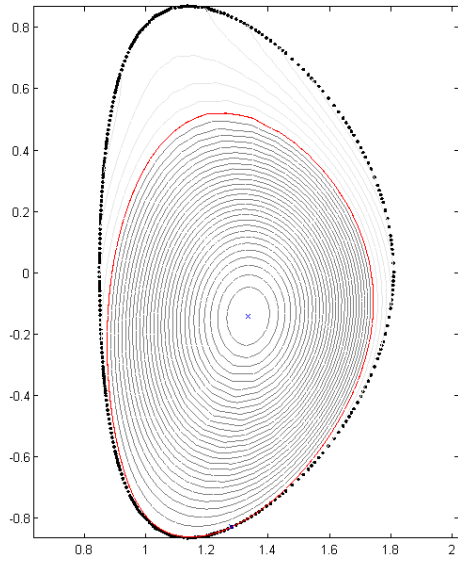
Mathematical and numerical formulation



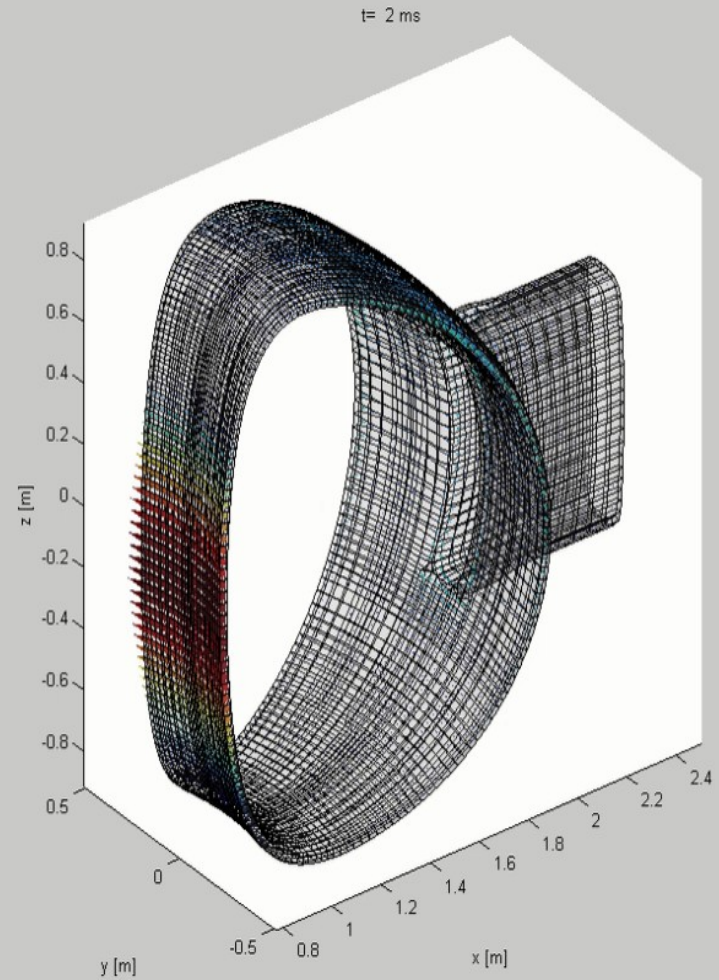
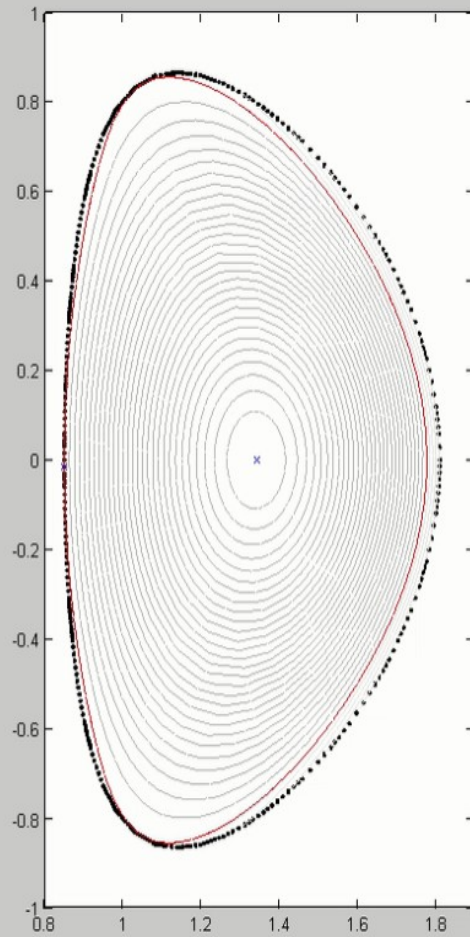
Examples of application

Conclusions

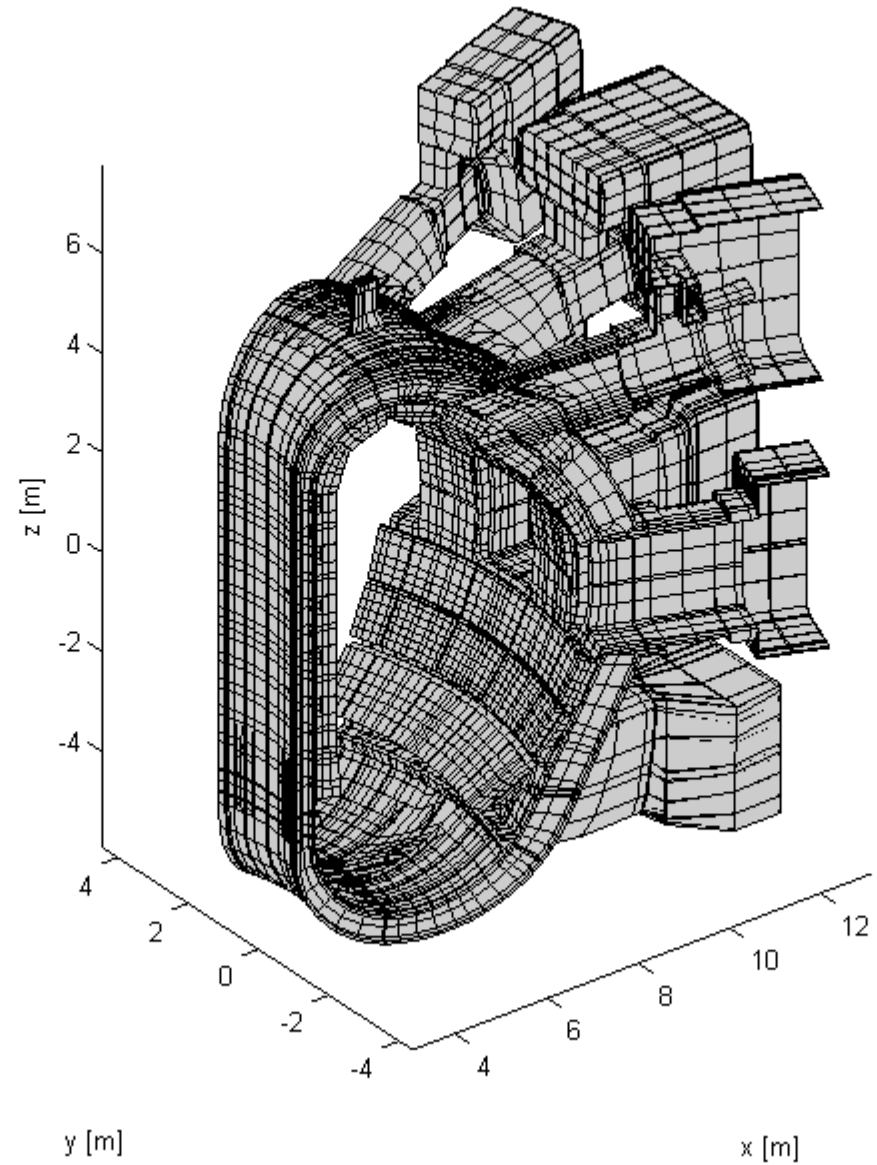
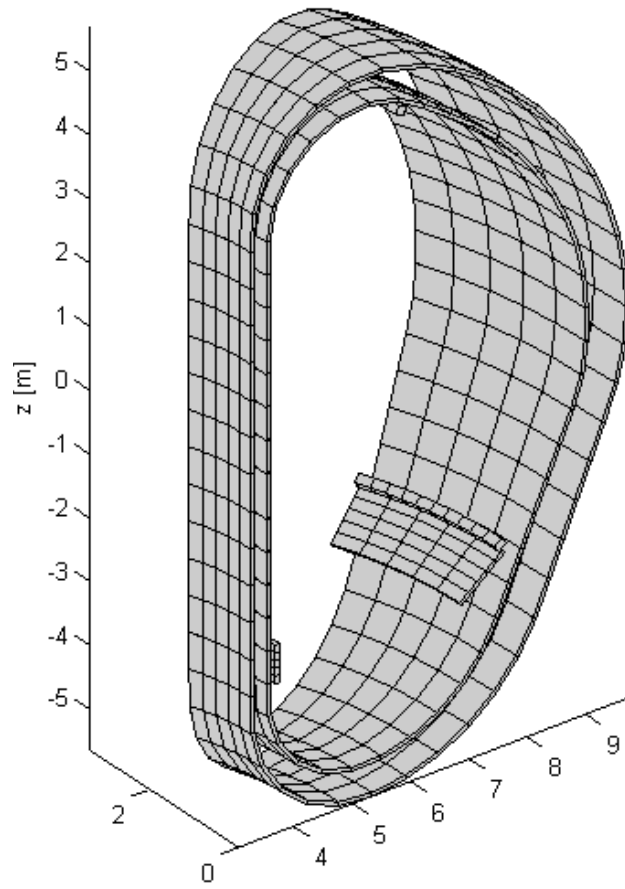
Application to IGNITOR



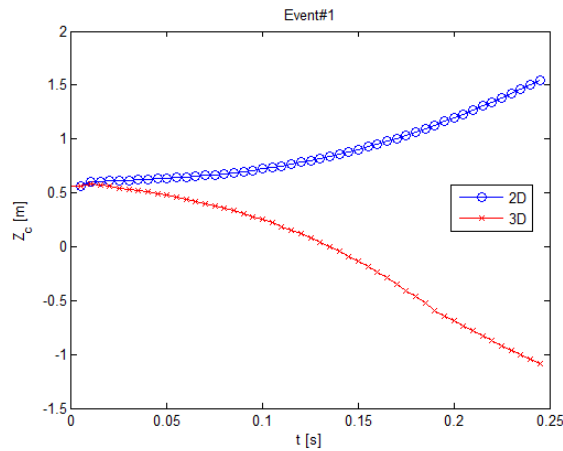
Application to IGNITOR



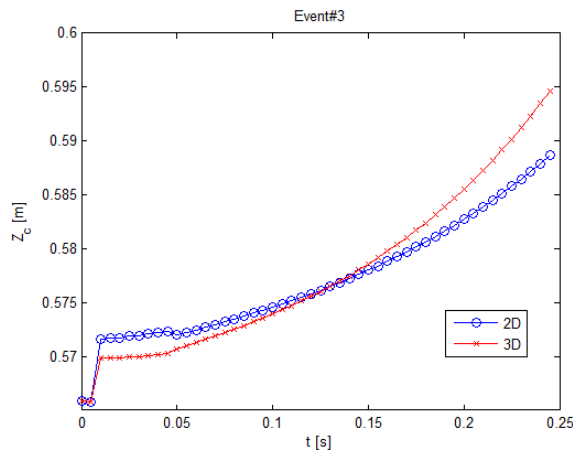
Application to ITER: 3D meshes



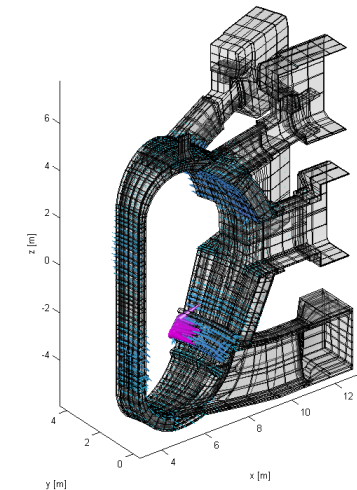
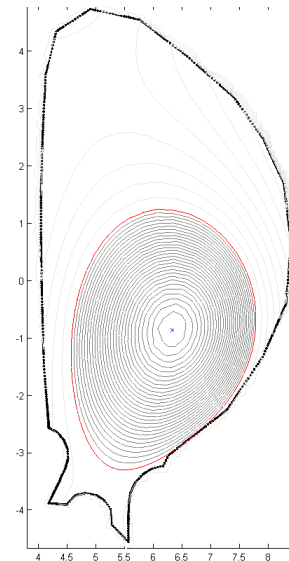
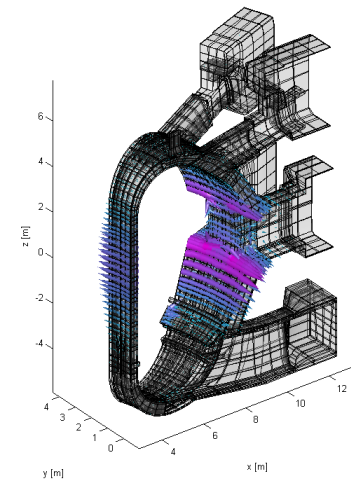
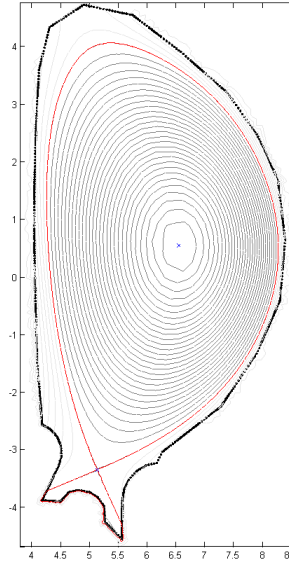
Application to ITER: 3D effects



In some cases 3D structures affect the direction of plasma vertical movement



In other cases 3D effects can be quantitatively significant (although with a qualitatively similar evolution)



ITER: Major disruption with upwards movement /1

Reference equilibrium: $I_p=15$ MA, $I_i=0.80$, $\beta_p=0.73$, centroid position (6.29,0.53) m

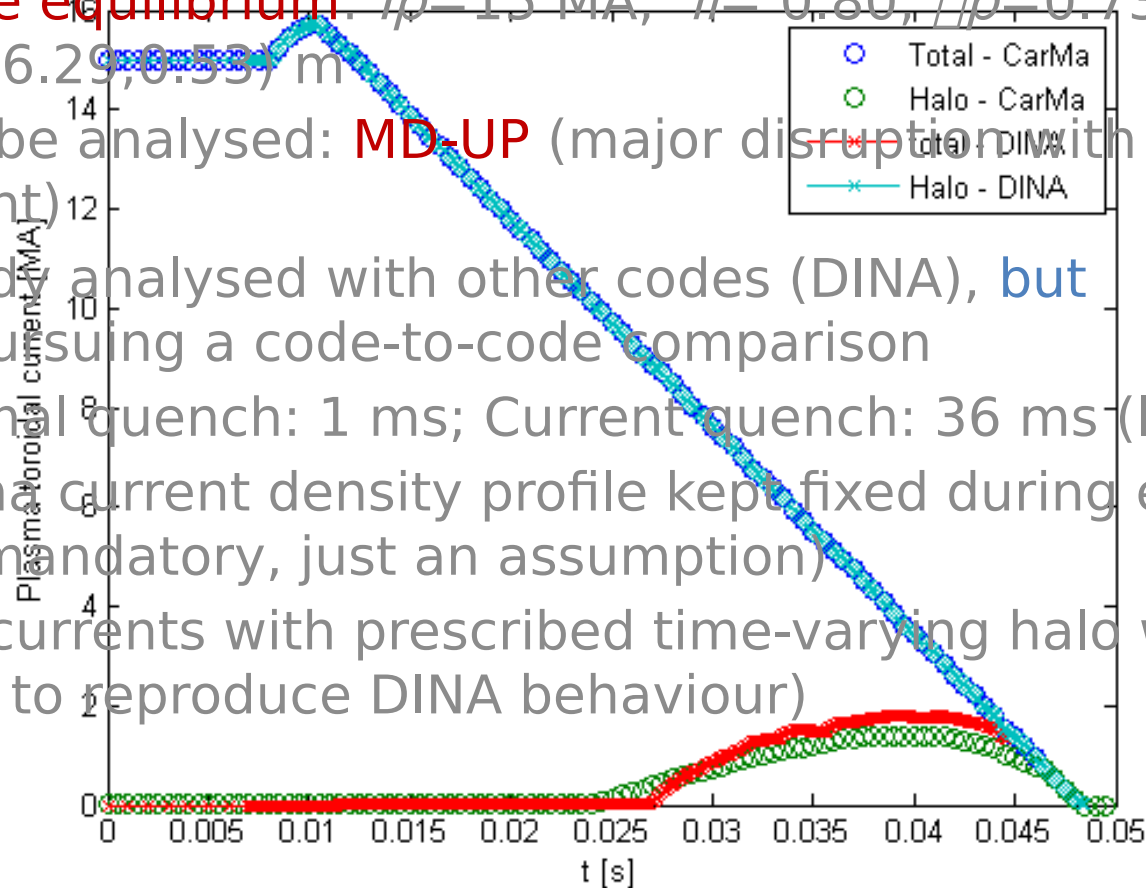
Event to be analysed: **MD-UP** (major disruption with upwards movement)

Already analysed with other codes (DINA), but not pursuing a code-to-code comparison

Thermal quench: 1 ms; Current quench: 36 ms (linear)

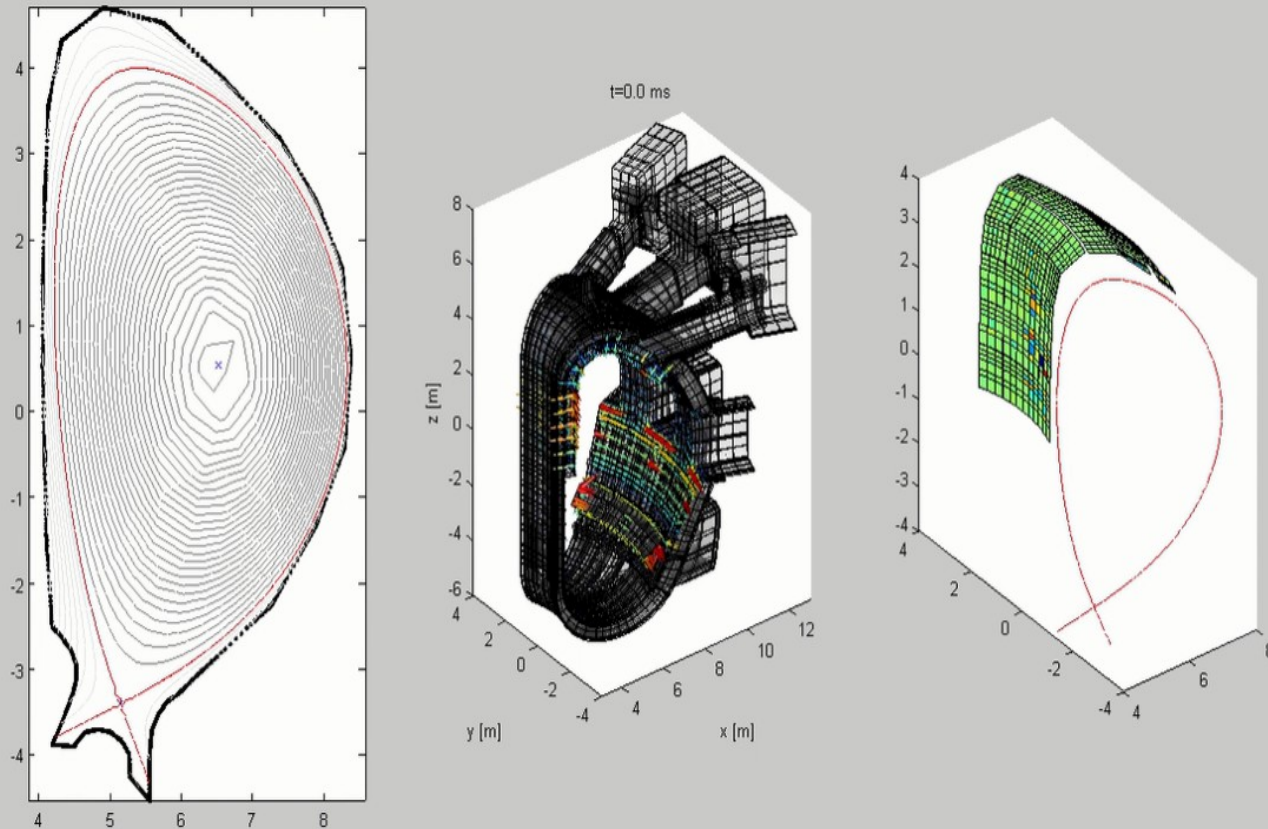
Plasma current density profile kept fixed during event (not mandatory, just an assumption)

Halo currents with prescribed time-varying halo width (no effort to reproduce DINA behaviour)



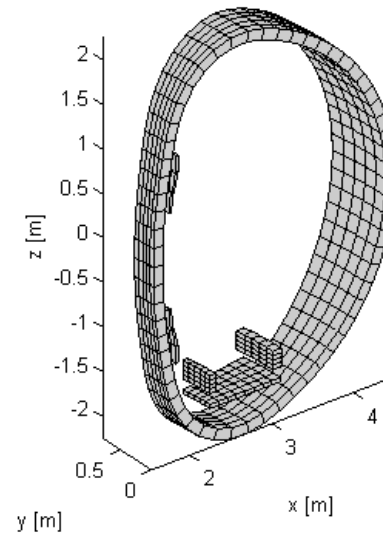
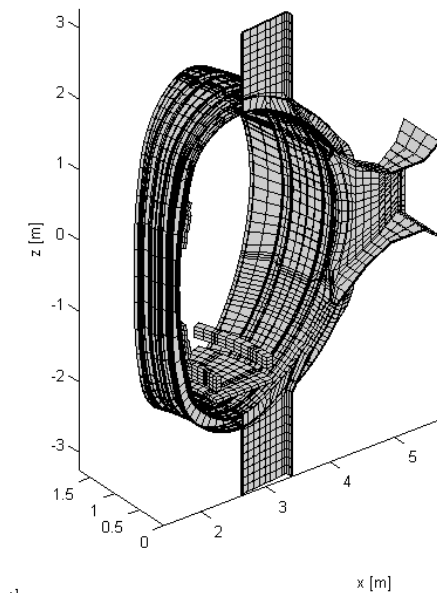
ITER: Major disruption with upwards movement /2

Self-consistent 3D nonlinear simulation with halo currents



Application to JET: Geometry

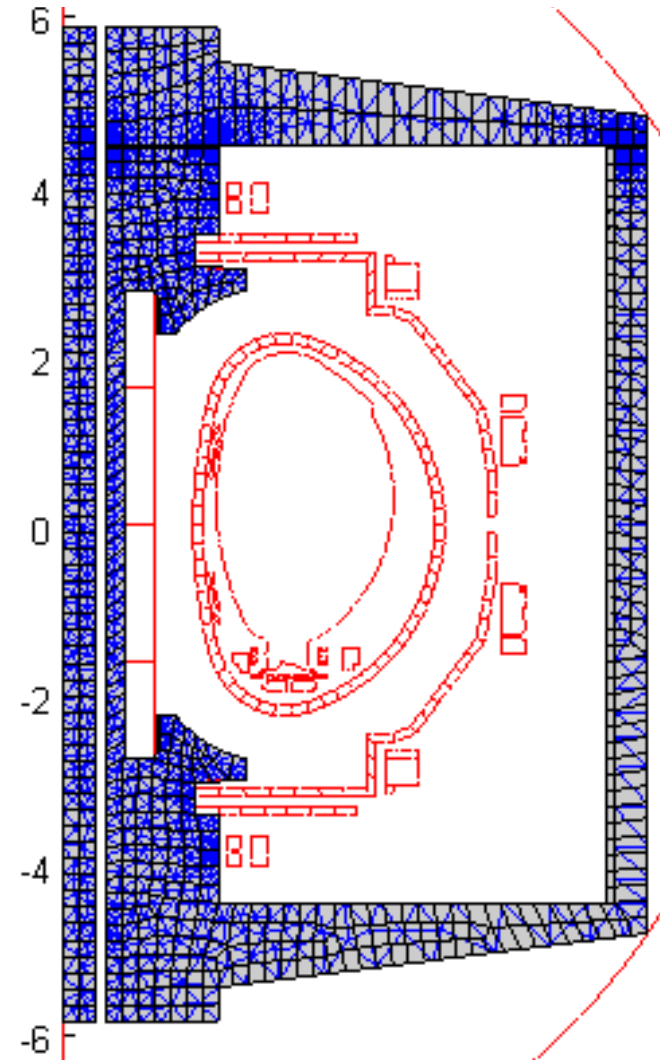
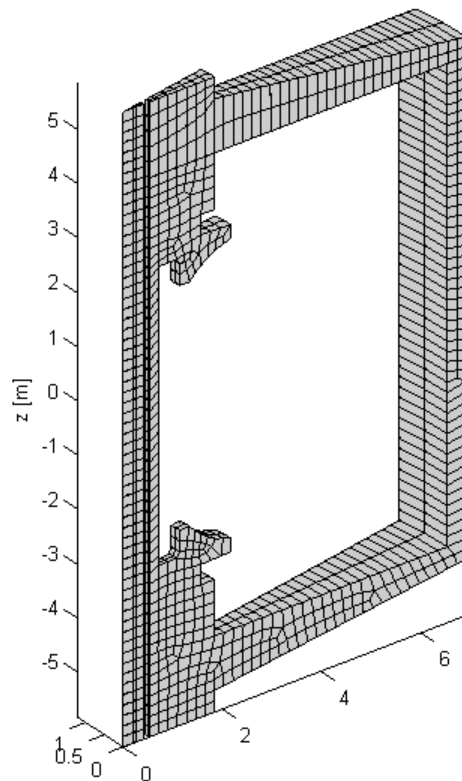
- ▶ 3D mesh of conducting structures
 - Mimicking an axisymmetric mesh
 - Fully 3D (ports, bellows, etc.)



Application to JET: Effect of magnetization

2D and 3D meshes

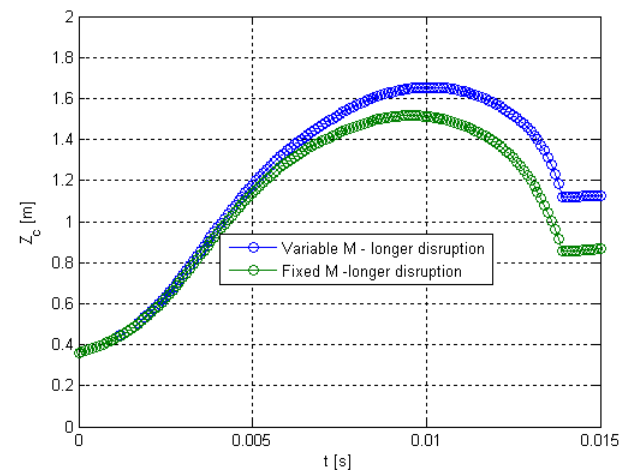
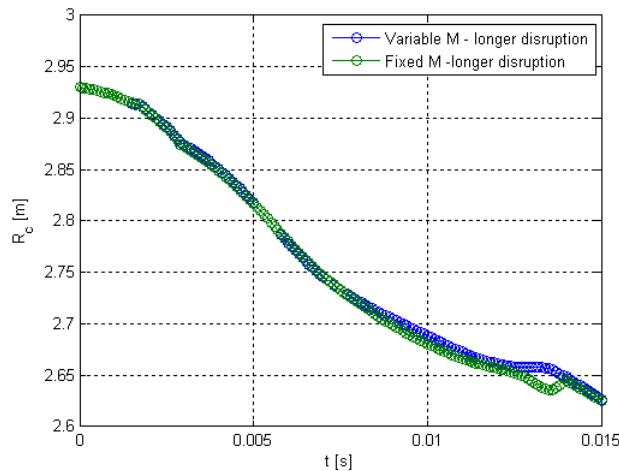
Identical geometry, but
2D mesh is finer



Application to JET: Effect of magnetization/2

Comparing fixed and variable magnetizations

As expected, for a slower disruption (current quench time 20 ms) the effect of variable magnetization is more noticeable (especially on vertical position)



Conclusions

Main achievements obtained by
developing effective 3D numerical models in the frame of the
computational electromagnetism
efficiently coupled to non nonlinear axisymmetric plasma evolution
The models have been validated it against benchmarks, experimental
data also in the frame of existing tokamaks

Other achievements not reported here

- Inclusion of feedback controller equations (for nonlinear/3D
validation of controllers designed on simplified linearized models)

Several further steps are ongoing

- Further developments of ferromagnetic materials (ferritic inserts,
etc.)

- Analysis of other plasma events where nonlinear/3D effects may be
important (e.g. minor disruptions, ramp up/down, AVDEs, L-H
transition, breakdown phase, ELMs etc.)

- Application to other devices (DEMO, IGNITOR, JT-60SA, RFX-mod,
EAST)

ACKNOWLEDGMENTS

This work was supported in part by the CREATE consortium and by Fusion for Energy

Magnetic Reconnection Sustained by the Thermonuclear Heating of the Electron Population

R. Gatto¹, B. Coppi²,

(1) Dept. of Energy Engineering, Sapienza University of Rome, Italy

(2) Dept. of Physics, Massachusetts Institute of Technology, Cambridge, MA, USA

Fisica del Plasma e Matematica Applicata

Il ruolo della matematica nello sviluppo della fisica del plasma

June 6, 2018, Aula Marconi, CNR Piazzale Aldo Moro, Roma

Endogenous modes associated with a finite electron temperature gradient can be sustained by the electron temperature heating rate due to the charged reaction products in a fusion burning plasma [1]. In this case, the longitudinal thermal conductivity on selected rational magnetic surfaces [1] is decreased, relative to its collisional value, by the effect of reconnection. If the relevant decrease is significant rational magnetic surfaces can acquire a favorite role in the process of plasma heating by fusion reaction products.

Supported in part by the U.S. DOE.

[1] B. Coppi, *et al.*, Nucl. Fusion, **55** 053011 (2015).

- TWO-FLUID ENERGY EQUATIONS
- IGNITION MODELS
- MODES AROUND A RATIONAL MAGNETIC SURFACE
- 1-D SLAB MODEL
- QUADRATIC FORM
- GROWTH RATE
 - with $(\tilde{T}_e, \tilde{B}_x) = (\text{even}, \text{odd})$
- PLASMA DISPLACEMENT
 - with $(\tilde{T}_e, \tilde{B}_x) = (\text{even}, \text{odd})$
- CURRENT/FUTURE WORK

TWO-FLUID ENERGY EQUATIONS

Neglecting stress tensor effects:

$$\begin{aligned}\frac{3}{2}n_e \left(\frac{\partial}{\partial t} + \mathbf{u}_e \cdot \nabla \right) T_e + p_e \nabla \cdot \mathbf{u}_e + \nabla \cdot \mathbf{q}_e &= S_e \\ \frac{3}{2}n_i \left(\frac{\partial}{\partial t} + \mathbf{u}_i \cdot \nabla \right) T_i + p_i \nabla \cdot \mathbf{u}_i + \nabla \cdot \mathbf{q}_i &= S_i\end{aligned}$$

with sources

$$\begin{aligned}S_e &= \frac{F_e^\alpha}{4} E_\alpha n_e^2 \langle \sigma v \rangle (T_i) + F_e^{\text{aux}} P_{\text{aux}} + \eta J^2 - C_B n_e^2 T_e^{1/2} - \frac{3}{2} n_e \bar{\nu}_{\text{eq}} (T_e - T_i) \\ S_i &= \frac{1 - F_e^\alpha}{4} E_\alpha n_e^2 \langle \sigma v \rangle (T_i) + (1 - F_e^{\text{aux}}) P_{\text{aux}} - \frac{3}{2} n_e \bar{\nu}_{\text{eq}} (T_i - T_e)\end{aligned}$$

Physical parameters (classical expressions)

- Electron collision frequency

$$\bar{\nu}_e = \bar{\nu}_{ei} + \bar{\nu}_{ee} \approx 2 \times \bar{\nu}_{ei} \equiv \overbrace{2 \times C_{ei}^\nu} \frac{n_e}{T_e^{3/2}}$$

- Ion collision frequency

$$\bar{\nu}_i = \bar{\nu}_{ii} = \sqrt{\frac{m_e}{m_i}} \bar{\nu}_{ei} \equiv \overbrace{\sqrt{\frac{m_e}{m_i}} C_{ei}^\nu} \frac{n_e}{T_e^{3/2}}$$

- Equilibration frequency

$$\bar{\nu}_{eq} = 2 \frac{m_e}{m_i} \bar{\nu}_{ei} \equiv C^{eq} \frac{n_e}{T_e^{3/2}}$$

- Resistivity

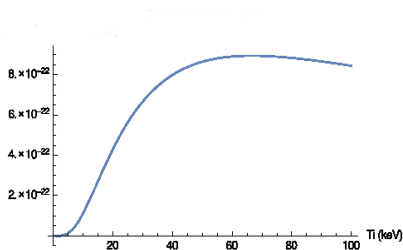
$$\eta = \frac{m_e \bar{\nu}_{ei}}{e^2 n_e} = \frac{C_{ei}^\nu m_e}{e^2 T_e^{3/2}}$$

DT “Reactivity”

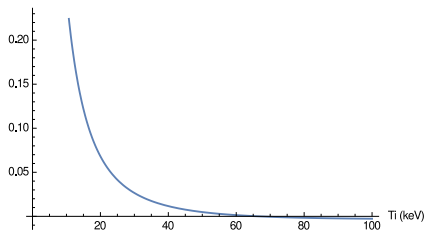
DT “Reactivity” in the **Bosh-Hale formulation** ($0.2 < T_i < 100$ keV)

$$\langle \sigma v \rangle = C_1 \theta(T_i) \sqrt{\frac{\xi}{(m_r c^2) T_i^3}} e^{-3\xi(T_i)}$$

$$\theta(T_i) = \frac{T_i}{1 - \frac{T_i[C_2 + T_i(C_4 + T_i C_6)]}{1 + T_i[C_3 + T_i(C_5 + T_i C_7)]}} \quad , \quad \xi(T_i) = \left(\frac{B_G^2}{4\theta(T_i)} \right)^{1/3}$$



$\langle \sigma v \rangle_{DT}$ vs T_i [m^3/s]



$(d\langle \sigma v \rangle_{DT}/dT_i)/\langle \sigma v \rangle_{DT}$ vs T_i [$1/keV$]

Heat transport:

- Electrons: parallel (along B) and perpendicular (radial) heat flux:

$$\mathbf{q}_e = \mathbf{q}_e^{\parallel} + \mathbf{q}_e^{\perp} = -n_e D_e^{\parallel} \nabla_{\parallel} T_e - n_e D_e^{\perp} \nabla_{\perp} T_e ,$$

- Ions: perpendicular (radial) heat flux:

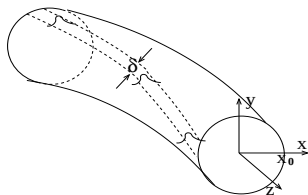
$$\mathbf{q}_i = \mathbf{q}_i^{\perp} = -n_i D_i^{\perp} \nabla_{\perp} T_i$$

- In 0-D models, any form of heating is delivered to the entire plasma volume
- In 1-1/2 models, heating depends on the radial coordinate only, and the influence of rational surfaces is neglected
- In reality, the existence of rational surfaces might have an important impact on the path to ignition, making it *an intrinsic 3-D phenomenon*

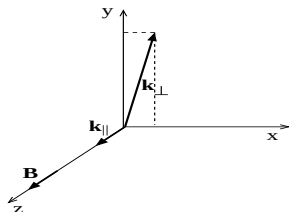
In the following we analyze the impact of rational surfaces on ignition condition

MODES AROUND RATIONAL MAGNETIC SURFACES

We perturb the equilibrium temperatures with a perturbation localized around the **rational surface** at $x = x_0$:



- $\delta \ll |\nabla \ln T_0|^{-1} \equiv L_{T_0}$



- $(k_{\perp})_x \simeq 0$, $(k_{\perp})_y \simeq k_{\perp}$

- $k_z = k_{\parallel} \ll k_{\perp}$

$\Rightarrow$ long wavelength modes along the magnetic field, radially localized within the scale-length δ

1-D SLAB MODEL

Considering a **1D slab model** ($x \leftrightarrow$ radial, $y \leftrightarrow$ poloidal, $z \leftrightarrow$ toroidal), neglecting ohmic heating, setting $F_e^{\text{aux}} = 1$, and **assuming constant heat diffusivities**:

$$\begin{aligned} \frac{3}{2} n_e \frac{\partial T_e}{\partial t} &+ \frac{3}{2} \mathbf{u}_{E,e} \cdot \nabla T_e - n_e D_e^{\parallel} \nabla \cdot \left(\hat{\mathbf{b}} \frac{\partial T_e}{\partial \ell_{\parallel}} \right) - n_e D_e^{\perp} \frac{\partial^2 T_e}{\partial x^2} \\ &= \frac{E_{\alpha}}{4} n_e^2 \langle \sigma v \rangle (T_i) - 3 n_e \frac{m_e}{m_i} \bar{\nu}_{ei} (T_e - T_i) - C_B n_e^2 T_e^{1/2} + P_{\text{aux}} \\ \frac{3}{2} n_i \frac{\partial T_i}{\partial t} &- n_i D_i^{\perp} \frac{\partial^2 T_i}{\partial x^2} = 3 n_e \frac{m_e}{m_i} \bar{\nu}_{ei} (T_e - T_i) \end{aligned}$$

where $\ell_{\parallel}$ is the coordinate along the magnetic field line, and $\hat{\mathbf{b}} \equiv \mathbf{B}/B$

- Linearize temperature, density and magnetic field:

$$Q(\mathbf{x}, t) \rightarrow Q(x) + \hat{Q}(\mathbf{x}, t)$$

where $Q(x)$ are the (global) equilibrium profiles, and

$$\hat{Q}(\mathbf{x}, t) = \tilde{Q}(x_{\perp}) e^{\gamma t} e^{ik_y y + ik_z z} \quad \text{with } \gamma = -i\omega = -i\omega_r + \omega_i$$

with $\tilde{Q}(x_{\perp} \equiv x - x_0)$ radially localized around $x = x_0$

- Taylor expand the “reactivity” and the Bremsstrahlung term around the equilibrium temperature:

$$\langle \sigma v \rangle(T_i) \rightarrow \langle \sigma v \rangle(T_i) + \left. \frac{d\langle \sigma v \rangle(T_i)}{dT_i} \right|_{T_i} \hat{T}_i$$

$$C_B n_e^2 T_e^{1/2} \rightarrow C_B n_e^2 T_e^{1/2} + \frac{C_B n_e^2}{2 T_e^{1/2}} \hat{T}_e$$

Parallel electron heat diffusion

Parallel electron heat conduction:

$$e^{ik_{\parallel}\ell_{\parallel}} \rightarrow \nabla \cdot \left(\hat{\mathbf{b}} \frac{\partial \hat{T}_e}{\partial \ell_{\parallel}} \right) = -k_{\parallel}^2 \hat{T}_e$$

where the **parallel wavevector** is expanded around the rational surface:

$$\begin{aligned} k_{\parallel} &= \frac{\mathbf{k} \cdot \mathbf{B}}{B} = \frac{k_y B_y(x) + k_z B_z}{B} \simeq \frac{1}{B} \left[\overbrace{k_y B_y(x_0) + k_z B_z}^{=0} + k_y \left. \frac{dB_y}{dx} \right|_{x_0} x_{\perp} \right] \\ &= \overbrace{\frac{k_y}{B} \left. \frac{dB_y}{dx} \right|_{x_0}}^{k'_{\parallel}} x_{\perp} \simeq \frac{k_y}{L_s} x_{\perp} \end{aligned}$$

Therefore:

$$\nabla \cdot \left(\hat{\mathbf{b}} \frac{\partial \hat{T}_e}{\partial \ell_{\parallel}} \right) \rightarrow -(k'_{\parallel})^2 x_{\perp}^2 \hat{T}_e \simeq -\frac{k_y^2}{L_s^2} x_{\perp}^2 \hat{T}_e$$

Linearized electron energy equation

Electron thermal energy balance keeping density, velocity and magnetic perturbations:

$$\begin{aligned} \frac{3}{2}\gamma \tilde{T}_e(x_\perp) + \frac{3}{2}\tilde{u}_{E,x}(x_\perp)\frac{dT_e}{dx} + D_e^\parallel(k_\parallel')^2 x_\perp^2 \tilde{T}_e(x_\perp) - ik_\parallel D_e^\parallel \frac{dT_e}{dx} \frac{\tilde{B}_x(x_\perp)}{B} \\ - D_e^\perp \frac{d^2 \tilde{T}_e(x_\perp)}{dx_\perp^2} = \gamma_{F0}^e \frac{\tilde{n}_e(x_\perp)}{n_e} + \gamma_F^e \tilde{T}_i(x_\perp) - \frac{3}{2}\nu_{eq}(T_e - T_i) \frac{\tilde{n}_e(x_\perp)}{n_e} \\ - \frac{3}{2}\nu_{eq}[\tilde{T}_e(x_\perp) - \tilde{T}_i(x_\perp)] \end{aligned}$$

where

$$\gamma_{F0}^e = \left. \frac{E_\alpha n_e \langle \sigma v \rangle}{2} \right|_{T_i}, \quad \gamma_F^e = \left. \frac{E_\alpha n_e}{4} \frac{d\langle \sigma v \rangle}{dT_i} \right|_{T_i}$$

Linearized ion energy equation

$$\frac{3}{2} \frac{\partial \tilde{T}_i(\mathbf{x}, t)}{\partial t} - D_i^\perp \frac{\partial^2 \tilde{T}_i(x_\perp)}{\partial x_\perp^2} = \frac{3C_e^\nu m_e n}{m_i T_e^{3/2}} [\tilde{T}_e(x_\perp) - \tilde{T}_i(x_\perp)]$$

To decouple the electron and ion equations, we approximate the ion perpendicular heat transport term as follows (δ_T = radial mode width):

$$-D_i^\perp \frac{d^2 \tilde{T}_i}{dx_\perp^2} \rightarrow +\nu_i^D \tilde{T}_i, \quad \nu_i^D = \text{const} \sim \frac{D_i^\perp}{\delta_T^2} \quad (1)$$

which leads, once introduced in the ion energy balance equation, to the proportionality relation

$$\tilde{T}_i = \epsilon_F \tilde{T}_e, \quad \epsilon_F = \frac{\nu_{eq}}{(3/2)\gamma + \nu_{eq} + \nu_i^D} = \text{const} \quad (2)$$

Electron energy equation

Using $\tilde{T}_i = \epsilon_F \tilde{T}_e$, defining

$$\bar{\gamma}_F^e \equiv -\gamma_F^e \epsilon_F + \frac{3}{2} \nu_{eq} (1 - \epsilon_F) + \frac{3}{2} \gamma, \quad \bar{\bar{\gamma}}_F^e \equiv -\bar{\gamma}_F^e$$

and using $k'_{\parallel} = k_y/L_s$ and $\bar{x}_{\perp} = (x - x_0)/\delta_T = x_{\perp}/\delta_T$, we rewrite the e-energy balance as:

$$\begin{aligned} \bar{\bar{\gamma}}_F^e \tilde{T}_e(\bar{x}_{\perp}) - D_e^{\parallel} (k'_{\parallel})^2 \bar{x}_{\perp}^2 \delta_T^2 \tilde{T}_e(\bar{x}_{\perp}) + i k'_{\parallel} \bar{x}_{\perp} \delta_T D_e^{\parallel} \frac{dT_e(x)}{dx} \frac{\tilde{B}_x(\bar{x}_{\perp})}{B} \\ + \frac{D_e^{\perp}}{\delta_T^2} \frac{d^2 \tilde{T}_e(\bar{x}_{\perp})}{d\bar{x}_{\perp}^2} - \frac{3}{2} \tilde{u}_{E,x} \frac{dT_e(x)}{dx} + \left[\gamma_{F0}^e - \frac{3}{2} \nu_{eq} (T_e - T_i) \right] \frac{\tilde{n}_e(\bar{x}_{\perp})}{n_e} = 0 \end{aligned}$$

with δ_T a (thermal) characteristic radial length of the inner region:

$$D_e^{\parallel} (k'_{\parallel})^2 \delta_T^2 \tilde{T}_e \sim \frac{D_e^{\perp}}{\delta_T^2} \tilde{T}_e \longrightarrow \delta_T \sim \left(\frac{D_e^{\perp}}{D_e^{\parallel}} \right)^{1/4} \left(\frac{L_s}{k_y} \right)^{1/2}$$

QUADRATIC FORM

Keep only the first four terms in the e-energy balance (i.e. neglect perturbed electron velocity and density) :

$$\begin{aligned}
 & \overset{\text{TERM (1)}}{\bar{\gamma}_F^e \tilde{T}_e(\bar{x}_\perp) - D_e^\parallel} \left[\overset{\text{TERM (2)}}{k_\parallel'^2 \bar{x}_\perp^2 \delta_T^2 \tilde{T}_e(\bar{x}_\perp) - ik_\parallel' \bar{x}_\perp \delta_T \frac{dT_e(x)}{dx} \frac{\tilde{B}_x(\bar{x}_\perp)}{B}} \right] \\
 & + \overset{\text{TERM (4)}}{\frac{D_e^\perp}{\delta_T^2} \frac{d^2 \tilde{T}_e(\bar{x}_\perp)}{d\bar{x}_\perp^2}} \simeq 0
 \end{aligned}$$

QUADRATIC FORM

Applying the operator $\int d\bar{x}_\perp \tilde{T}_e^*(\bar{x}_\perp)$ to the four terms, and using in TERM ③ the infinite parallel conductivity e-energy balance relation, $\tilde{T}_e^*(\bar{x}_\perp) \simeq (1/ik'_\parallel \bar{x}_\perp \delta_T)(dT_e(x)/dx)(\tilde{B}_x^*(\bar{x}_\perp)/B)$, the e-energy balance leads to the quadratic form ($\langle \dots \rangle \equiv \int d\bar{x}_\perp \dots$):

$$\begin{aligned} \bar{\gamma}_F^e \langle |\tilde{T}_e(\bar{x}_\perp)|^2 \rangle &= D_e^\parallel \left[k_\parallel'^2 \delta_T^2 \langle \bar{x}_\perp^2 |\tilde{T}_e(\bar{x}_\perp)|^2 \rangle - \left(\frac{dT_e(x)}{dx} \right)^2 \left\langle \left| \frac{\tilde{B}_x(\bar{x}_\perp)}{B} \right|^2 \right\rangle \right] \\ &- \frac{D_e^\perp}{\delta_T^2} \left\langle \left| \frac{d\tilde{T}_e(\bar{x}_\perp)}{d\bar{x}_\perp} \right|^2 \right\rangle \simeq 0 \end{aligned}$$

where we recall that

$$\bar{\gamma}_F^e = \gamma_F^e \epsilon_F - \frac{3}{2} \nu_{eq} (1 - \epsilon_F) - \frac{3}{2} \gamma, \quad \epsilon_F = \frac{\nu_{eq}}{(3/2)\gamma + \nu_{eq} + \nu_i^D}$$

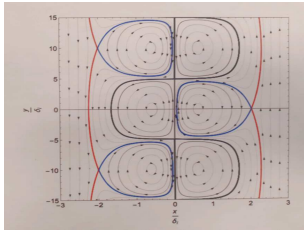
NOTE: the parallel conductivity term is reduced by the magnetic perturbation

Consequences on the onset of the thermonuclear instability:

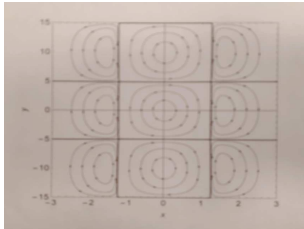
- heating on irrational magnetic surfaces is rapidly spread over the entire surface
- without magnetic perturbation, heating on rational magnetic surface is rapidly spread over the finite length of the closed magnetic field line (formation of hot helical filaments)
- in the presence of magnetic islands, parallel heat conduction on closed field line is decreased, and *the thermonuclear instability is favored*

If the parallel heat conduction is decreased enough by magnetic perturbations, rational surfaces might become the location where ignition initiates

QUADRATIC FORM



Odd-parity magnetic perturbation



Plasma flow

GROWTH RATE WITH $(\tilde{T}_e, \tilde{B}_x) = (\text{EVEN}, \text{ODD})$

Assume $\tilde{T}_e$ even and $\tilde{B}_x$ odd:

$$\tilde{T}_e(\bar{x}_\perp) = \tilde{T}_e e^{-\sigma \bar{x}_\perp^2/2}, \quad \tilde{B}_x(\bar{x}_\perp) = -i \tilde{B}_x \bar{x}_\perp e^{-\sigma \bar{x}_\perp^2/2}$$

where $\tilde{T}_e, \tilde{B}_x$ are constant amplitudes

Inserting the expressions for $\tilde{T}_e(\bar{x}_\perp)$ and $\tilde{B}_x(\bar{x}_\perp)$ in electron thermal energy balance:

$$\left(-\bar{\gamma}_F^e + \frac{D_e^\perp}{\delta_T^2} \sigma \right) \tilde{T}_e + \left[\left(D_e^\parallel \frac{k_y^2}{L_s^2} \delta_T^2 - \frac{D_e^\perp}{\delta_T^2} \sigma^2 \right) \tilde{T}_e - \frac{k_y}{L_s} \delta_T D_e^\parallel \frac{dT_e}{dx} \frac{\tilde{B}_x}{B} \right] \bar{x}_\perp^2 \simeq 0$$

Setting equal to zero the first and the second term separately, solving for σ , and equating:

$$\left[\frac{(\bar{\gamma}_F^e)^2}{(D_e^*)^2} \Delta_s^4 - 1 \right] \delta_T = -\frac{\tilde{B}_x}{B} \frac{dT_e/dx}{\tilde{T}_e} \Delta_s^2$$

where $D_e^* \equiv (D_e^\parallel D_e^\perp)^{1/2}$ and $\Delta_s \equiv (L_s/k_y)^{1/2}$

GROWTH RATE WITH $(\tilde{T}_e, \tilde{B}_x) = (\text{EVEN}, \text{ODD})$

Solving for $\overline{\gamma}_F^e = \gamma_F^e \epsilon_F - \frac{3}{2} \nu_{eq} (1 - \epsilon_F) - \frac{3}{2} \gamma$:

$$\overline{\gamma}_F^e = \frac{D_e^*}{\Delta_s^2} \sqrt{1 - \frac{(dT_e/dx) \Delta_s^2}{\tilde{T}_e \delta_T} \frac{\tilde{B}_x}{B}}$$

Using $\epsilon_F = \nu_{eq} / [(3/2)\gamma + \nu_{eq} + \nu_i^D]$ we obtain a quadratic equation for γ with solutions:

$$\gamma = \frac{1}{2} \left\{ - \left(\frac{5}{3} \nu_{eq} + \frac{2}{3} \nu_i^D + \frac{2D_e^* \sqrt{R(\tilde{B}_x)}}{3\Delta_s^2} \right) \pm \sqrt{\left(\frac{5}{3} \nu_{eq} + \frac{2}{3} \nu_i^D + \frac{2D_e^* \sqrt{R(\tilde{B}_x)}}{3\Delta_s^2} \right)^2 - 4 \left[\nu_{eq} \left(\frac{2}{3} \nu_i^D - \frac{4}{9} \gamma_F^e \right) + \frac{4D_e^* \sqrt{R(\tilde{B}_x)}}{9\Delta_s^2} (\nu_{eq} + \nu_i^D) \right]} \right\}$$

where we have defined the quantity:

$$R(\tilde{B}_x) \equiv 1 - \frac{(dT_e/dx) \Delta_s^2}{\tilde{T}_e \delta_T} \frac{\tilde{B}_x}{B}$$

GROWTH RATE WITH $(\tilde{T}_e, \tilde{B}_x) = (\text{EVEN}, \text{ODD})$

- Case $R(\tilde{B}_x) > 0$: Instability condition is:

$$\left(\frac{5}{3} \nu_{eq} + \frac{2}{3} \nu_i^D + \frac{2D_e^* \sqrt{R(\tilde{B}_x)}}{3\Delta_s^2} \right)^2 < 4 \left[\nu_{eq} \left(\frac{2}{3} \nu_i^D - \frac{4}{9} \gamma_F^e \right) + \frac{4D_e^* \sqrt{R(\tilde{B}_x)}}{9\Delta_s^2} (\nu_{eq} + \nu_i^D) \right]$$

- Case $R(\tilde{B}_x) = -|R(\tilde{B}_x)| < 0$: Define:

$$a \equiv \left(\frac{5}{3} \nu_{eq} + \frac{2}{3} \nu_i^D \right)^2 - \left(\frac{2D_e^* \sqrt{|R(\tilde{B}_x)|}}{3\Delta_s^2} \right)^2 - 4\nu_{eq} \left(\frac{2}{3} \nu_i^D - \frac{4}{9} \gamma_F^e \right)$$

$$b \equiv \frac{4D_e^* \sqrt{|R(\tilde{B}_x)|}}{9\Delta_s^2} (\nu_{eq} + \nu_i^D)$$

The instability condition is:

$$-\frac{2}{3} \frac{D_e^*}{\Delta_s^2} \sqrt{|R(\tilde{B}_x)|} + \frac{\sqrt{2}}{2} \sqrt{\sqrt{a^2 + b^2} - a} > 0$$

PLASMA DISPLACEMENT

To get an expression for the displacement eigenfunction, we couple the *electron thermal energy balance* with the *longitudinal electron momentum*.

- For the latter we adopt the following form:

$$0 \simeq -\frac{\widehat{\mathbf{B} \cdot \nabla p_e}}{B} - en_e \left(\hat{E}_{\parallel} - \eta_{\parallel} \hat{J}_{\parallel} - \mathcal{L}_I \frac{\partial \hat{J}_{\parallel}}{\partial t} \right) + e\eta_{\parallel} J_z(x) \hat{n}_e$$

where we have assumed no equilibrium electric field, and included the inductive electric field.

- Using $\widehat{\mathbf{B} \cdot \nabla p_e}/B = ik_{\parallel} \hat{p}_e + (\hat{B}_x/B)(dp_e(x)/dx)$ and $\hat{p}_e = \hat{n}_e T_e(x) + n_e(x) \hat{T}_e$:

$$ik_{\parallel} \hat{T}_e + \left(\frac{dT_e}{dx} + T_e \frac{dn_e/dx}{n_e} \right) \frac{\hat{B}_x}{B} + e \left[\hat{E}_{\parallel} + (i\omega \mathcal{L}_I - \eta_{\parallel}) \hat{J}_{\parallel} \right] + [ik_{\parallel} T_e - e\eta_{\parallel} J_z(x)] \frac{\hat{n}_e}{n_e} = 0$$

PLASMA DISPLACEMENT

From Maxwell equations $\hat{\mathbf{B}} = \nabla \times \hat{\mathbf{A}}$, $\hat{\mathbf{E}} = -\nabla \hat{\Phi} - (1/c)(\partial \hat{\mathbf{A}}/\partial t)$, $\hat{\mathbf{J}} = (c/4\pi)\nabla \times \hat{\mathbf{B}}$, and assuming $\hat{\mathbf{A}} \simeq \hat{A}_z \mathbf{e}_z$:

$$\hat{B}_x \simeq ik_y \hat{A}_z, \quad \hat{B}_y \simeq -d\hat{A}_z/dx, \quad \hat{B}_z \simeq 0$$

$$\hat{E}_x \simeq -d\hat{\Phi}/dx, \quad \hat{E}_y \simeq -ik_y \hat{\Phi}, \quad \hat{E}_z \simeq -ik_z \hat{\Phi} + i\omega \hat{A}_z/c$$

Parallel electric field:

$$\hat{E}_{\parallel} \simeq \hat{E}_z \simeq -ik_z \hat{\Phi} + \frac{\omega/k_y}{c} \hat{B}_x$$

Parallel current density:

$$\hat{J}_{\parallel} \simeq \hat{J}_z \simeq \frac{c}{4\pi} \left(\frac{d\hat{B}_y}{dx} - ik_y \hat{B}_x \right) \simeq \frac{ic}{4\pi k_y} \left(\frac{d^2 \hat{B}_x}{dx^2} - k_y^2 \hat{B}_x \right) \simeq \frac{ic}{4\pi k_y} \frac{d^2 \hat{B}_x}{dx^2}$$

where we have assumed $k_y^2 \ll |\partial^2/\partial x^2|$

PLASMA DISPLACEMENT

Assuming $\hat{\mathbf{u}}_e \simeq \hat{\mathbf{u}}_E = c(\hat{\mathbf{E}} \times \mathbf{B})/B^2$ (no equilibrium electric field):

$$\begin{aligned}\hat{u}_x &\simeq -ick_y \hat{\Phi}/B + ick_z (B_y/B) \hat{\Phi} - (\omega/k_y)(B_y/B)(\hat{B}_x/B) \simeq -ick_y \hat{\Phi}/B, \\ \hat{u}_y &\simeq (c/B)(d\hat{\Phi}/dx), \quad \hat{u}_z \simeq -(cB_y/B^2)(d\hat{\Phi}/dx)\end{aligned}$$

with $\nabla \cdot \hat{\mathbf{u}}_e \simeq 0$.

Plasma displacement:

$$\hat{u}_x = -i\omega \hat{\xi}_x \longrightarrow \hat{\Phi} \simeq (B\omega/ck_y)\hat{\xi}_x \text{ and } \hat{E}_{\parallel} \simeq (\omega/ck_y) [\hat{B}_x - ik_{\parallel} B \hat{\xi}_x]$$

Continuity equation:

$$\hat{n}_e \simeq -\frac{dn_e}{dx} \hat{\xi}_x - i \frac{ck_{\parallel}}{4\pi e\omega k_y} \frac{d^2 \hat{B}_x}{dx^2} \simeq -\frac{dn_e}{dx} \hat{\xi}_x$$

PLASMA DISPLACEMENT

Using all these results, the *longitudinal electron momentum balance* becomes:

$$ik_{\parallel} \hat{T}_e + \frac{dT_e}{dx} \frac{\hat{B}_x}{B} + T_e \frac{dn_e/dx}{n_e} \left(\frac{\hat{B}_x}{B} - ik_{\parallel} \hat{\xi}_x \right) + e \left[\frac{\omega}{ck_y} (\hat{B}_x - ik_{\parallel} B \hat{\xi}_x) + (i\omega \mathcal{L}_I - \eta_{\parallel}) \frac{ic}{4\pi k_y} \frac{d^2 \hat{B}_x}{dx^2} \right] + e\eta_{\parallel} J_z(x) \frac{dn_e/dx}{n_e} \hat{\xi}_x = 0$$

Using the *electron thermal energy balance* to eliminate the first two terms we obtain a single equation in $\hat{T}_e, \hat{\xi}_x, \hat{B}_x$:

$$\begin{aligned} -\frac{\bar{\gamma}_F^e k_y}{ik_{\parallel}} \frac{D_B}{D_{\parallel}^e} \frac{\hat{T}_e}{T_e} - \frac{k_y}{ik_{\parallel}} \frac{D_B D_{\perp}^e}{D_{\parallel}^e} \frac{1}{T_e} \frac{d^2 \hat{T}_e}{dx_{\perp}^2} + \left[-ik_{\parallel} (\omega - \omega_{*e}) + \frac{3\omega}{2} \frac{\omega_{*T}}{k_{\parallel} D_{\perp}^e} \right. \\ \left. - [\gamma_F^e - \frac{3}{2} \nu_{eq} (T_e - T_i)] \frac{\omega_{*e}}{ik_{\parallel} D_{\perp}^e T_e} \right] \hat{\xi}_x + (\omega - \omega_{*e}) \frac{\hat{B}_x}{B} - \left(\frac{\omega d_I^2}{B} + \frac{ic^2 \eta_{\parallel}}{4\pi B} \right) \frac{d^2 \hat{B}_x}{dx^2} \\ - \omega_{*e} \left(\frac{e\eta_{\parallel} J_z}{T_e} - ik_{\parallel} \right) \hat{\xi}_x = 0 \end{aligned}$$

where we have used $\mathcal{L}_I = 4\pi d_I^2/c^2$ where d_I is the “inductive skin depth”.

PLASMA DISPLACEMENT WITH $(\hat{T}_e, \hat{B}_x) = (\text{even}, \text{odd})$

Assume $\tilde{T}_e$ even and $\tilde{B}_x$ odd:

$$\tilde{T}_e(\bar{x}_\perp) = \tilde{T}_e e^{\sigma \bar{x}_\perp^2 / 2}, \quad \tilde{B}_x(\bar{x}_\perp) = -i \tilde{B}_x \bar{x}_\perp e^{\sigma \bar{x}_\perp^2 / 2}$$

Introducing into a simplified version of the equation:

$$(\omega - \omega_{*e}) \frac{\hat{B}_x}{B} - \frac{\omega d_I^2}{B \delta_T^2} \frac{d^2 \hat{B}_x}{d \bar{x}_\perp^2} - i k_\parallel (\omega - \omega_{*e}) \hat{\xi}_x - \frac{\bar{\gamma}_F^e k_y}{i k_\parallel} \frac{D_B}{D_\parallel^e} \frac{\hat{T}_e}{T_e} = 0$$

and solving for the displacement:

$$\begin{aligned} \hat{\xi}_x(\bar{x}_\perp) = & - \frac{e^{\sigma \bar{x}_\perp^2 / 2}}{\delta_T (k_y / L_s) (\omega - \omega_{*e})} \\ & \times \left\{ \left[\omega - \omega_{*e} - \frac{\omega d_I^2}{\delta_T^2} (\bar{x}_\perp^2 \sigma^2 - 3\sigma) \right] \frac{\tilde{B}_x}{B} - \frac{\bar{\gamma}_F^e k_y}{\delta_T (k_y / L_s) \bar{x}_\perp} \frac{D_B}{D_\parallel^e} \frac{\tilde{T}_e}{T_e} \right\} \end{aligned}$$

- A model for parallel heat diffusivity in presence of magnetic reconnection
- Quantitative discussion of the instability condition
- Derivation of instability condition from an improved fluid model [improved treatment of e-i coupling; maintain density and velocity perturbations (as done for plasma displacement); relation between thermal and inductive characteristic lengths; etc.]
- Inclusion of toroidal effects
- ...